Assignment 5

Probability Theory II
(EN.553.721, Spring 2026)

Assigned: April 21, 2026 Due: 11:59pm EST, May 1, 2026

Submit solutions in LaTeX. Write in complete sentences. Include and justify all steps of your arguments, but avoid writing excessive explanation that is not contributing to your solution.

See my website for policies about late submissions, collaboration, and use of AI assistants.

Problem 1 (Brownian bridge)**.** Let $B(t)$ be a Brownian motion. Define $R(t) := B(t) - tB(1)$. Note that $R(0) = R(1) = 0$. Intuitively, think of $R(t)$ as a Brownian motion "pinned" to take value 0 at time 1 (see also the comment about this below).

1. Show that $R(t)$ is a Gaussian process (i.e., that all of its finite marginal distributions are Gaussian random vectors). Compute its mean function $\mu(t) = \mathbb{E}[R(t)]$ and covariance kernel $K(s,t) = \mathrm{Cov}[R(s), R(t)]$. You may but are not required to check that, confirming the above intuition, for any $0 \le t_1 < \cdots < t_k < 1$, the law of $(R(t_1), \ldots, R(t_k), R(1) = 0)$ is the same as that of $(B(t_1), \ldots, B(t_k), B(1))$ conditional on $B(1) = 0$. This can be done using the calculations about conditioning Gaussian vectors from Homework 2.

2. Let $X_1, X_2, \ldots$ be i.i.d. random variables that have distribution function $F(t) := \mathbb{P}[X_i \le t]$. You may assume that F is strictly increasing. Note that F is a function $F : \mathbb{R} \to [0,1]$. Let $F_n : \mathbb{R} \to [0,1]$ be the empirical distribution function of n i.i.d. samples:
$$F_n(t) := \frac{\#\{i \in \{1, \ldots, n\} : X_i \le t\}}{n}.$$
This is a random function, and thus we may view it as a continuous-time random process. Define the associated normalized process
$$\widehat{F}_n(t) := \sqrt{n} \cdot (F_n(t) - F(t)).$$
Define a final process $X(t) := R(F(t))$, a "time change" of $R(t)$. Show that $(\widehat{F}_n(t))_{t \in \mathbb{R}}$ converges in finite marginal distributions to $(X(t))_{t \in \mathbb{R}}$, i.e., that for any $t_1 < \cdots < t_k$, we have
$$(\widehat{F}_n(t_1), \ldots, \widehat{F}_n(t_k)) \Rightarrow (X(t_1), \ldots, X(t_k)).$$
(**Hint:** Show that a vector of the form $(\widehat{F}_n(t_1), \ldots, \widehat{F}_n(t_k))$ may be viewed as a normalized sum of n i.i.d. random vectors having some covariance matrix $\boldsymbol{\Sigma} \in \mathbb{R}^{k \times k}$. Use the multidimensional central limit theorem on this sum.)

Problem 2 (A first stochastic integral)**.** Let $B(t)$ be a Brownian motion.

1. Show that, almost surely,

$$\lim_{n\to\infty} \sum_{j=1}^{2^n} \left| B\left(\frac{j}{2^n}\right) - B\left(\frac{j-1}{2^n}\right) \right| = \infty.$$

2. Show that, almost surely,

$$\lim_{n\to\infty} \sum_{j=1}^{2^n} \left| B\left(\frac{j}{2^n}\right) - B\left(\frac{j-1}{2^n}\right) \right|^2 = 1.$$

3. Let $F : [0,1] \to \mathbb{R}$ be a smooth function. For each $n \geq 0$, define

$$I_n := \sum_{j=1}^{2^n} 2^n \cdot \left(F\left(\frac{j}{2^n}\right) - F\left(\frac{j-1}{2^n}\right)\right) \cdot \left(B\left(\frac{j}{2^n}\right) - B\left(\frac{j-1}{2^n}\right)\right).$$

   Show that the I_n converge in L^2 to some (random) limit. Intuitively, you should view this limit as $\int_0^1 F' \, dB$, a stochastic Riemann-Stieltjes integral.

   (**Hint:** Show that the I_n form a martingale and use a suitable convergence theorem.)