Assignment 4

Probability Theory II
(EN.553.721, Spring 2026)

Assigned: April 6, 2026 Due: 11:59pm EST, April 21, 2026

Submit solutions in LaTeX. Write in complete sentences. Include and justify all steps of your arguments, but avoid writing excessive explanation that is not contributing to your solution.

See my website for policies about late submissions, collaboration, and use of AI assistants.

Problem 1 (Harmonic analysis and Markov chains)**.** Let (Y_n) be a Markov chain taking values in a countable S and initialized from some probability measure μ_0 over S with transition kernel $p : S \times S \to [0, 1]$. Suppose that S is irreducible.

1. Let $f : S \to \mathbb{R}$ satisfy, for all $y \in S$,

$$f(y) \geq \sum_{z \in S} p(y, z) f(z) = \mathop{\mathbb{E}}_{y} f(Y_1)$$

   and have that $f(Y_n) \in L^1$ for all $n \geq 0$. Show that $(f(Y_n))_{n \geq 0}$ is a supermartingale, a submartingale if the inequality above is reversed, and a martingale if the inequality is an equality. In these cases, f is called *superharmonic*, *subharmonic*, and *harmonic*, respectively. For $U \subseteq S$, write $\partial U = \{y \notin U : p(x, y) > 0 \text{ for some } x \in U\}$. Then, $f : U \cup \partial U \to \mathbb{R}$ is called *super/sub/harmonic on* U if it satisfies the above for all $y \in U$ (a weaker condition than having these properties on all of S).

2. Let $U \subseteq S$ be finite, non-empty, and not equal to all of S. Suppose $f : U \cup \partial U \to \mathbb{R}$ is harmonic on U. Show that

$$\max_{x \in U} f(x) \leq \sup_{x \in \partial U} f(x).$$

3. Again, let $U \subseteq S$ be finite, non-empty, and not all of S. For any $f : \partial U \to \mathbb{R}$, show that there is at most one $g : U \cup \partial U \to \mathbb{R}$ that is harmonic on U and that satisfies the "boundary condition" $g(x) = f(x)$ for all $x \in \partial U$.

   (Hint: If g and h are two different such functions, consider using Part 2 on $g - h$ and $h - g$.)

4. Once more, let $U \subseteq S$ be finite, non-empty, and not all of S. Let $T := \min\{n \geq 0 : Y_n \notin U\}$ be the *exit time* from U. Suppose that $f : \partial U \to \mathbb{R}$ is bounded. Show that the function

$$g(x) := \mathop{\mathbb{E}}_{x} f(Y_T)$$

is well-defined for $x \in U \cup \partial U$, is harmonic on U, and satisfies $g(x) = f(x)$ for all $x \in \partial U$. That is, averaging the value of f at an exit time gives the unique solution of the associated "boundary value problem" from Part 3.

If you are familiar with basic notions of continuous harmonic analysis and/or the Poisson partial differential equation, you should compare Parts 2 and 3 to the corresponding analytic principles.

Problem 2 (Recurrence and transience of trees)**.** Let G_d be the infinite d-regular tree, the infinite tree where every vertex has degree $d \geq 2$. (For example, $d = 2$ gives the integer graph $\mathbb{Z}$ we have discussed in class.) The *simple random walk (SRW)* on a tree is the Markov chain taking values in the tree where each successive state is a uniformly random neighbor of the current state.

1. Fix some $x \in G_d$. Let W_n be the number of closed walks in G_d that start at x, follow an edge at every step, traverse a total of n edges, and end back at x. Show that $W_n = 0$ if n is odd. If n is even, let $W_{n,k}$ be the number of such walks that visit x exactly k times (note that we must have $k \geq 2$). Show that $W_{n,k} \leq 2^{n-k+1} d^{k-1} (d-1)^{n/2-k+1}$.

   (HINT: Group the steps of a walk into three types: (1) steps from x to one of its neighbors, (2) steps from vertices other than x that increase the distance to x, and (3) steps from vertices other than x that decrease the distance to x. Show that the subsequence of Type (2) and (3) steps determines where the Type (1) steps must have been. Bound the number of possible patterns of step types and the number of walks with each pattern.)

2. Derive that, if $d \geq 3$, then there is a constant $C_d > 0$ such that $W_n \leq C_d (2\sqrt{d-1})^n$. Conclude that all $x \in G_d$ are transient for the SRW on G_d for any $d \geq 3$. (In contrast, by Pólya's theorem from class, all $x \in G_2 = \mathbb{Z}$ are recurrent for the SRW on G_2.)

3. Consider $G_2 = \mathbb{Z}$. In class we showed that the SRW on $\mathbb{Z}$ is recurrent. Consider now picking some $\beta \in (\frac{1}{2}, 1)$, and taking a Markov chain on $\mathbb{Z}$ with the "biased" transition kernel
$$p(x, y) = \left\{ \begin{array}{ll} \beta & \text{if } y = x + 1, \\ 1 - \beta & \text{if } y = x - 1, \\ 0 & \text{otherwise} \end{array} \right\}.$$
   Show that every $x \in \mathbb{Z}$ is transient for a Markov chain with this transition kernel with any initialization.

Problem 3 (More on stationary measures)**.** Let (Y_n) be a Markov chain on a countable S such that S is irreducible and every $x \in S$ is recurrent. Recall that we studied the stationary measure $\mu_x(y) := \mathbb{E}_x[\sum_{n=0}^{T_x - 1} \mathbb{1}\{Y_n = y\}]$, where $T_x := \min\{n \geq 1 : Y_n = x\}$.

1. We say that a random variable A *stochastically dominates* a random variable B if, for all $t \in \mathbb{R}$, $\mathbb{P}[A \geq t] \geq \mathbb{P}[B \geq t]$. Write $A \succeq B$ for this relation (not to be confused with positive semidefiniteness of matrices). Show that, if $A \succeq B$, and $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing, then $\mathbb{E}f(A) \geq \mathbb{E}f(B)$ provided that both expectations are well-defined.

2. Show that, for all $x, y \in S$ distinct, $\mathbb{P}_y[T_x < T_y] > 0$.

3. Conclude that $\sum_{n=0}^{T_x - 1} \mathbb{1}\{Y_n = y\}$ is stochastically dominated by a suitable geometric random variable, and therefore that $\mu_x(y) < \infty$ for all $x, y \in S$.

4. Suppose that, for some $x \in S$, $\mathbb{E}_x[T_x] < \infty$. Show that then in fact $\mathbb{E}_y[T_y] < \infty$ for all $y \in S$. Generally, such states in a Markov chain are called *positive recurrent*, and positive recurrence, like recurrence, is "contagious under accessibility" (but you do not need to prove that beyond this special case). You may use the result of Part 3 that the μ_x are measures on S, and the uniqueness theorem that shows all stationary measures are the same up to constant rescaling.