Assignment 3

Probability Theory II
(EN.553.721, Spring 2026)

Assigned: March 9, 2026   Due: 11:59pm EST, March 27, 2026

Submit solutions in LaTeX. Write in complete sentences. Include and justify all steps of your arguments, but avoid writing excessive explanation that is not contributing to your solution.

See my website for policies about late submissions, collaboration, and use of AI assistants.

Problem 1 (Martingales for random walks)**.** Let $X_i \sim \mathrm{Unif}(\{\pm 1\})$ be i.i.d., and let $S_n = \sum_{i=1}^n X_i$ with $S_0 = 0$ be the usual simple random walk.

1. Let $T := \min\{n : S_n = a\}$ be the hitting time of some $a > 0$. We showed in class using the optional stopping theorem that $\mathbb{E}T = \infty$. Show that $T < \infty$ almost surely.

   (**Hint:** Show the convergence almost surely of a suitable martingale.)

2. Let $\lambda \in \mathbb{R}$ and
$$M_n := \frac{\exp(\lambda S_n)}{\cosh(\lambda)^n}.$$
   Show that M_n is a martingale.

3. Using the optional stopping theorem, calculate $f(z) := \mathbb{E}z^T$ over $z \in [-1, 1]$.

4. When $a = 1$, derive a closed formula for $\mathbb{P}[T = k]$ for all $k \geq 0$ by computing a Taylor series of $f(z)$. In particular, you may look up and use Newton's generalized binomial series. Your formula should only involve arithmetic and binomial coefficients. Extend your result to $a > 1$, giving a formula in terms of sums of binomial coefficients. You may but are not required to simplify this general formula to a closed form as simple as the $a = 1$ case.

Problem 2 (Time to observe a sequence)**.** Let $a_1, \dots, a_N \in \{0, 1\}$ be fixed, and let $X_1, X_2, \dots \sim \mathrm{Unif}(\{0, 1\})$ be i.i.d. In this problem, you will study the time required to see the sequence $(a_1, \dots, a_N)$ in the random sequence $(X_1, X_2, \dots)$:

$$T := \min\{n : n \geq N, X_n = a_N, X_{n-1} = a_{N-1}, \dots, X_{n-N+1} = a_1\}.$$

Consider a Casino where a countable number of Gamblers $1, 2, \dots$ bet on the outcome of the X_n at each time n. Gambler i bets $B_{i,n,0} \in L^1$ on the outcome being $X_n = 0$ at time n, and

$B_{i,n,1} \in L^1$ on the outcome being $X_n = 1$ at time n. At each time n, almost surely at most K_n Gamblers make a non-zero bet, for some deterministic $K_n \in \mathbb{Z}_{\geq 0}$. If $X_n = 0$, then the Casino's fortune increases by $B_{i,n,1}$ for each i and decreases by $B_{i,n,0}$ for each i, and Gambler i's fortune increases by $B_{i,n,0}$ and decreases by $B_{i,n,1}$; if $X_n = 1$, the same happens with 0 and 1 switched. Write $(G_{i,n})_{n\geq 0}$ for the fortune of Gambler i at each time.

1. Suppose that the processes $(B_{i,n,s})_n$ for each $i \geq 1$, $s \in \{0,1\}$ are L^1 and predictable with respect to $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$. Define the net profit of the Casino from the bets made at time n as:

$$Y_0 := 0,$$
$$Y_n := \sum_{i\geq 1} \mathbb{1}\{X_n = 0\}(B_{i,n,1} - B_{i,n,0}) + \sum_{i\geq 1} \mathbb{1}\{X_n = 1\}(B_{i,n,0} - B_{i,n,1}).$$

   Define the net profit up to time n as:

$$M_n := \sum_{j=1}^{n} Y_j.$$

   Show that M_n is a martingale.

2. The Gamblers play strategies similar to the martingale betting strategy from class. Each Gambler starts with a fortune of $G_{i,0} = 1$. Gambler 1 bets all of their money on $X_i = a_i$, until either losing and stopping or leaving with a fortune of 2^N:

$$B_{1,n,s} := \mathbb{1}\{1 \leq n \leq N, X_1 = a_1, \dots, X_{n-1} = a_{n-1}, s = a_n\} \cdot 2^{n-1}.$$

   The other Gamblers $j \geq 2$ play the same strategy, but starting at time j:

$$B_{j,n,s} := \mathbb{1}\{j \leq n \leq N + j - 1, X_j = a_1, \dots, X_{n-1} = a_{n-j}, s = a_{n-j+1}\} \cdot 2^{n-j}.$$

   Suppose that $N = 2$ and $(a_1, a_2) = (0, 1)$. Describe M_T in terms of T for each $T \geq 2$. Compute $\mathbb{E}[M_T]$ using the optional stopping theorem. Use that to compute $\mathbb{E}[T]$.

3. Repeat Part 2 for $N = 6$ and $(a_1, a_2, a_3, a_4, a_5, a_6) = (1, 1, 0, 0, 1, 1)$.

4. Describe a general formula for $\mathbb{E}[T]$ in terms of $(a_1, \dots, a_N)$. For a given length N, find a sequence that makes $\mathbb{E}[T]$ as large as possible and another that makes $\mathbb{E}[T]$ as small as possible.

Problem 3 (Product martingales). Let $X_1, X_2, \dots$ be independent random variables such that $X_i \geq 0$ almost surely and $\mathbb{E}X_i = 1$ for all i. We have seen that

$$M_0 := 1,$$
$$M_n := \prod_{i=1}^{n} X_i \text{ for } n \geq 1$$

defines a martingale. Define $s_i := \mathbb{E}\sqrt{X_i}$.

1. Show that $0 < s_i \leq 1$ for all i.

2. Show that there exists a random variable $M_\infty \geq 0$ such that $M_n \to M_\infty$ almost surely.

3. Show that if $\prod_{i=1}^\infty s_i = 0$, then $M_\infty = 0$ almost surely.

   (**Hint:** Consider another martingale formed by suitably normalizing $\prod_{i=1}^n \sqrt{X_i}$.)

4. Show that if $\prod_{i=1}^\infty s_i > 0$, then $M_n \to M_\infty$ in L^1, and therefore $\mathbb{E}M_\infty = 1$ and it is not the case that $M_\infty = 0$ almost surely.

   (**Hint:** Use the martingale from Part 3 and a suitable martingale maximal inequality to show $\sup_n M_n < \infty$ almost surely. Then, apply dominated convergence.)

Problem 4 (Sums of random length)**.** Let $X_1, X_2, \dots$ be i.i.d. L^1 random variables and let $S_n := \sum_{i=1}^n X_i$. Let $N \in \mathbb{Z}_{\geq 0}$ be a random variable independent of the X_i having $N \in L^1$.

1. Show that $\mathbb{E}S_N = \mathbb{E}\sum_{i=1}^N X_i = \mathbb{E}N \cdot \mathbb{E}X_1$.

   (**Hint:** Write $S_N = \sum_{i=1}^\infty X_i \mathbb{1}\{i \leq N\}$. Handle the case $X_i \geq 0$ first.)

2. Show that, if further $X_i \in L^2$, then $\mathbb{E}(S_N - N\mathbb{E}X_1)^2 = \mathbb{E}N \cdot \mathrm{Var}[X_1]$ and conclude that, if also $N \in L^2$, then $\mathbb{E}S_N^2 = \mathbb{E}N \cdot \mathbb{E}X_1^2 + \mathbb{E}[N(N-1)] \cdot (\mathbb{E}X_1)^2$ and therefore also $\mathrm{Var}[S_N] = \mathrm{Var}[N] \cdot (\mathbb{E}X_1)^2 + \mathbb{E}N \cdot \mathrm{Var}[X_1]$.

   (**Hint:** Reduce to the case $\mathbb{E}X_1 = 0$. Use the L^2 martingale convergence theorem.)

3. Calculate the mean and variance of the sum of the rolls of a six-sided die made until a 6 is rolled, not including the roll where the 6 appears.

Problem 5 (Concentration inequalities)**.** This problem will explore a class of concentration inequalities proved using the Azuma and McDiarmid inequalities.

1. Let $X_1, X_2, \dots, X_n$ be independent random variables taking values in $\mathbb{R}^d$ equipped with a norm $\|\cdot\|$ (not necessarily the Euclidean norm—look up the axioms a norm must satisfy if you are not familiar) such that $\|X_i\| \leq 1$ almost surely for all $i \in [n]$. Show that there is an absolute constant $c > 0$ such that

$$\mathbb{P}\left[\left|\left\|\sum_{i=1}^n X_i\right\| - \mathbb{E}\left\|\sum_{i=1}^n X_i\right\|\right| > t\right] \leq 2\exp\left(-\frac{ct^2}{n}\right).$$

2. Let $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n \in \mathbb{R}^d$ having $\|\boldsymbol{v}_i\| \leq 1$ be deterministic and let $z_1, \dots, z_n \sim \mathrm{Unif}(\{\pm 1\})$. Show that there are absolute constants $c, C > 0$ such that

$$\mathbb{P}\left[\left\|\sum_{i=1}^n z_i \boldsymbol{v}_i\right\| > (C+t)\sqrt{n}\right] \leq 2\exp\left(-ct^2\right).$$

   Here $\|\cdot\|$ is the ordinary Euclidean norm.

3. Let $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ be a random matrix (not necessarily symmetric) having independent entries such that $|A_{ij}| \le 1$ almost surely for all $i, j \in [n]$. Show that

$$\mathbb{P}[|\|\boldsymbol{A}\|_{\mathsf{op}} - \mathbb{E}\|\boldsymbol{A}\|_{\mathsf{op}}| > t] \le 2 \exp\left(-\frac{ct^2}{n}\right).$$

Here $\|\cdot\|_{\mathsf{op}}$ is the operator norm of matrices, also equal to the largest singular value.

(**Hint:** Consider different ways of grouping the entries of the matrix.)