Assignment 3

Random Matrix Theory in Data Science and Statistics

(EN.553.796, Fall 2026)

Assigned: October 22, 2026 Due: November 6, 2026, 11:59PM

Solve all problems. Problems 2–4 are worth an equal amount towards your homework grade. Problem 1 counts towards your final project proposal grade.

Submit solutions in LaTeX. Write in complete sentences. Include and justify all steps of your arguments, but avoid writing excessive discussion that is not contributing to your solution. You are welcome to include images if you think that will help explain your solutions.

Problem 1 (Project proposal)**.** Propose a paper or collection of related papers to read, a computational experiment to perform, or an open problem to study for your final project. See the course website for ideas. Write 2-3 paragraphs here doing one of the following:

- If you plan to read a paper, tell us the title and author(s), read the abstract and introduction, and describe how it relates to the course, what aspect of the paper you are interested in, and what else you might have to read or do to understand it.
- If you plan to run a computational experiment, describe the experiment, give one or two relevant references that might help set your expectations for what you will find, and explain what the experiment will tell you about random matrices.
- If you plan to try working on an open problem, write down the problem, give a few references concerning it, and outline in one paragraph what approach you plan to try or what experiments you can perform.

In all cases, your final project will consist of a short presentation at the end of the class (10-15 minutes) and a short write-up about whatever you choose here.

Problem 2 (Tail estimates for the semicircle law)**.** Suppose that ν has mean 0, variance 1, and all moments finite. Let $\boldsymbol{W}^{(d)} \sim \mathsf{Wig}(d, \nu)$ and write $\lambda_1 \geq \cdots \geq \lambda_d$ for the eigenvalues of $\frac{1}{\sqrt{d}}\boldsymbol{W}^{(d)}$. You may use the convergence of expected moments proved in class: for every integer $k \geq 1$,

$$\lim_{d\to\infty} \mathbb{E}\left[\frac{1}{d}\sum_{i=1}^{d} \lambda_i^k\right] = \int x^k \, d\mu_{\mathsf{SC}}(x).$$

This problem asks you to control the contribution of eigenvalues outside a neighborhood of the semicircle support to general averages as on the left-hand side.

1. For $\epsilon > 0$, let $N_d(\epsilon) := \#\{i : |\lambda_i| \geq 2 + \epsilon\}$. Show that, for every integer $\ell \geq 1$,

$$N_d(\epsilon) \leq \frac{1}{(2+\epsilon)^{2\ell}} \sum_{i=1}^{d} \lambda_i^{2\ell}.$$

   Conclude that

$$\frac{1}{d} N_d(\epsilon) \xrightarrow{\mathbb{P}} 0$$

   as $d \to \infty$ for every fixed $\epsilon > 0$, and therefore that also

$$\frac{1}{d} \sum_{i:|\lambda_i|>2+\epsilon} |f(\lambda_i)| \xrightarrow{\mathbb{P}} 0$$

   for every fixed bounded continuous function $f : \mathbb{R} \to \mathbb{R}$ and $\epsilon > 0$.

   (**Hint:** Use Markov's inequality on $\mathbb{P}[N_d(\epsilon) \geq \delta d]$ for any constant $\delta > 0$, then use convergence of expected moments and a suitable bound on the Catalan numbers. First take $d \to \infty$ with ℓ fixed, then take $\ell \to \infty$.)

2. Adapt the argument from Part 1 to show that the latter conclusion also holds for polynomials: for every fixed polynomial p and fixed $\epsilon > 0$,

$$\frac{1}{d} \sum_{i:|\lambda_i|>2+\epsilon} |p(\lambda_i)| \xrightarrow{\mathbb{P}} 0.$$

   (**Hint:** Reduce to monomials $p(x) = x^k$. Show that

$$\frac{1}{d} \sum_{i:|\lambda_i|>2+\epsilon} |\lambda_i|^k \leq \frac{1}{d(2+\epsilon)^{2\ell-k}} \sum_{i=1}^{d} \lambda_i^{2\ell}$$

   for $2\ell > k$. Apply Markov's inequality and convergence of expected moments, taking the limits in the same order as in Part 1.)

Problem 3 (Semicircle law with only two moments)**.** In the same setting as Problem 2, suppose that ν has mean 0 and variance 1, with no further moment assumptions. Prove that, for every smooth and compactly supported function $f : \mathbb{R} \to \mathbb{R}$,

$$\frac{1}{d}\sum_{i=1}^{d} f(\lambda_i) \xrightarrow{\mathbb{P}} \int f \, d\mu_{\mathsf{SC}} \tag{1}$$

as $d \to \infty$. You may use that the same result holds under the additional assumption that all moments of the entry distribution are finite, as we sketched in lecture.

(**Hint:** For a large constant C, truncate the off-diagonal entries by setting

$$T_{ij} := W_{ij}^{(d)} \mathbb{1}\{|W_{ij}^{(d)}| \le C\},$$

then center and normalize them:

$$\widetilde{W}_{ij}^{(d,C)} := \frac{T_{ij} - \mathbb{E}[T_{ij}]}{\sqrt{\mathsf{Var}[T_{ij}]}}.$$

Keep the diagonal zero and make the matrix symmetric. Apply the all-moments semicircle theorem to this matrix. Use the Hoffman–Wielandt inequality and its consequence for spectral averages from Homework 2, Problem 3, to compare it with the original matrix. First take $d \to \infty$ with C fixed, then take $C \to \infty$.)

Problem 4 (Heavy tails and the spectral norm)**.** This problem clarifies some details of the convergence of the extreme eigenvalues of Wigner matrices.

1. Suppose that $\boldsymbol{W}^{(d)}$ are Wigner matrices and $\lambda_i = \lambda_i^{(d)}$ are the eigenvalues of $\frac{1}{\sqrt{d}}\boldsymbol{W}^{(d)}$, and suppose that the convergence in (1) holds for all f smooth and compactly supported. Show that, for any fixed $\epsilon > 0$,

$$\lim_{d\to\infty} \mathbb{P}[\lambda_1 \le 2 - \epsilon] = 0.$$

2. Find a fixed probability distribution ν with mean 0 and variance 1 such that, for $\boldsymbol{W}^{(d)} \sim \mathsf{Wig}(d, \nu)$, there is a deterministic sequence $C_d \to \infty$ with

$$\lim_{d\to\infty} \mathbb{P}\left[\left\|\frac{1}{\sqrt{d}}\boldsymbol{W}^{(d)}\right\| \ge C_d\right] = 1.$$

   Here $\|\cdot\|$ denotes the operator norm. Thus, even though the empirical spectral distribution converges to the semicircle law (by Problem 3), the extreme eigenvalues can escape arbitrarily far from its support.

   (**Hint:** Prove and use that $\|\boldsymbol{W}\| \ge \max_{i,j\in[d]} |W_{ij}|$. Choose a heavy-tailed distribution with finite variance, and estimate the probability that all of the independent upper-triangular entries stay below a given threshold.)