(10/13/26) Semicircle Limit Thm, Continued

Thm: $W = W^{(d)} \sim \mathrm{Wig}(0,d)$, $\hat{W} = \frac{1}{\sqrt{d}} W$

$\forall f: \mathbb{R} \to \mathbb{R}$ bdd cts, $\frac{1}{d}\sum_i f(\lambda_i(\hat{W}))$

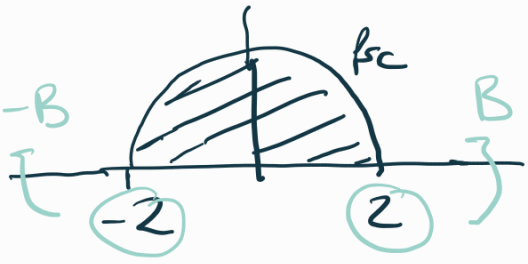
$\downarrow$ (WP)

$\int_{-2}^{2} f(x)\rho_{sc}(x)\,dx$

Last time:

Lem: $\forall p \in \mathbb{R}[x]$, $\mathbb{E}\,\frac{1}{d}\sum_i p(\lambda_i(\hat{W})) \to \int_{-2}^{2} p(x)\rho_{sc}(x)\,dx$.

Idea of Thm from Lem:



$\rightsquigarrow$ Expect all $\lambda_i(\hat{W}) \in [-2-o(1), 2+o(1)]$

$\hookleftarrow$ all $|\lambda_i(\hat{W})| \le 2+o(1)$.

Fix $B > 2$. Weierstrass approx for f on $[-B,B]$:

$\exists\, p(x) \in \mathbb{R}[x]$ s.t. $|p(x)-f(x)| \le \varepsilon \quad \forall x \in [-B,B]$.

Use to control

$$\mathbb{P}\left[\left|\frac{1}{d}\sum f(\lambda_i) - \int f\rho_{sc}\right| > \delta\right] \overset{(?)}{\to} 0.$$

($\lambda_i = \lambda_i(\hat{W})$, random var.)

by triangle ineq. to produce $|p-f|$:

$$\left| \frac{1}{d} \sum f(\lambda_i) - \int f \, d\rho_{sc} \right|$$

Lem $\Rightarrow \to 0$, $\leq \varepsilon$ for d large

$$\leq \left| \mathbb{E} \frac{1}{d} \sum p(\lambda_i) - \int_{-2}^{2} p \rho_{sc} \right| + \left| \frac{1}{d} \sum p(\lambda_i) - \mathbb{E} \frac{1}{d} \sum p(\lambda_i) \right| \quad (1)$$

$$+ \left| \frac{1}{d} \sum_{|\lambda_i| \leq B} (f(\lambda_i) - p(\lambda_i)) \right| + \frac{1}{d} \sum_{|\lambda_i| > B} |f(\lambda_i)| + \frac{1}{d} \sum_{|\lambda_i| > B} |p(\lambda_i)|$$

$\sim \leq \varepsilon$ (2) (3)

$$+ \left| \int_{-2}^{2} (f - p) \rho_{sc} \right| \sim \leq \varepsilon$$

$$\leq 3\varepsilon + (1) + (2) + (3)$$

$$\mathbb{E}(1)^2 \overset{\text{(Jensen)}}{\leq} \mathbb{E} \left(\frac{1}{d} \sum p(\lambda_i) - \mathbb{E} \frac{1}{d} \sum p(\lambda_i) \right)^2$$

$$= \mathrm{Var} \left[\frac{1}{d} \sum p(\lambda_i) \right] = \mathrm{Var} \left[\frac{1}{d} \mathrm{Tr}(\hat{W}^k) \right].$$

Claim: this $\mathrm{Var} \xrightarrow[d \uparrow \infty]{} 0$.

Enough to show for $p(x) = x^k$.

More complicated version of Lem (see notes + exercises):

$$(1) \xrightarrow{(L^2)} 0 \implies (1) \xrightarrow{(P)} 0$$

For (2): f bdd, say $|f(x)| \leq C \ \forall x$

$\Rightarrow$ (2) $\le C \cdot \underbrace{\frac{\#\{i \in [d] : |\lambda_i| > B\}}{d}}_{\text{fraction of } \lambda_i \text{ s.t. } |\lambda_i| > B}$

("outlier" eigvals).

Trick: $\frac{1}{d}\#\{|\lambda_i| > B\} = \frac{1}{d}\sum_{i=1}^{d} \mathbb{1}\{|\lambda_i| > B\}$

$\le \frac{1}{d}\sum_{i=1}^{d} \frac{\lambda_i^{2\ell}}{B^{2\ell}} = \frac{1}{B^{2\ell}} \cdot \boxed{\frac{1}{d}\mathrm{Tr}(\hat{W}^{2\ell})}$

$\mathbb{E}\,\frac{1}{d}\#\{|\lambda_i| > B\} \le \mathbb{E}\{\cdot\} \xrightarrow[d\uparrow\infty]{} \frac{C_\ell}{B^{2\ell}} \le \left(\frac{2}{B}\right)^{2\ell}$ (C_ℓ: Catalan)

Recall: $C_\ell = \frac{1}{\ell+1}\binom{2\ell}{\ell} \le 2^{2\ell}$ ↑

$\ell \uparrow \infty \Rightarrow$ (2) $\xrightarrow{(1)} 0 \Rightarrow$ (1) $\xrightarrow{(2)} 0$.

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(3): slight generalization of same idea

Combining $\rightsquigarrow$ pf of Thm.

Main tasks beyond finding $\lim_{d\uparrow\infty} \mathbb{E}\frac{1}{d}\mathrm{Tr}(\hat{W}^4)$:

(1) Concentration of $\frac{1}{d}\mathrm{Tr}(\hat{W}^4)$ (eg. $\mathrm{Var} \to 0$)

(2) Control of # + size of outlier eigenvalues.

Extra variants of semicircle thm?

① Universality: same limit for any choice of entry distribution W_{ij} (W_{ij} "$=$" W_{ji})

$W_{ij} \overset{i.i.d}{\sim} \boxed{\nu}$ fixed as $d \to \infty$

$\to \mathbb{E} W_{ij} = 0$

$\to \mathrm{Var}[W_{ij}] = \mathbb{E} W_{ij}^2 = 1$

$\to$ "regularity" $\mathbb{E}|W_{ij}|^4 < \infty$.

More generally, sometimes but not always true.

Ex: Consider W_{ij} i.i.d., $= \begin{cases} +1/\sqrt{p} & \text{w/p } p/2 \\ 0 & \text{w/p } 1-p \\ -1/\sqrt{p} & \text{w/p } p/2 \end{cases}$

$\mathbb{E} W_{ij} = 0$, $\mathbb{E} W_{ij}^2 = \frac{1}{p} \cdot p = 1$.

p const. $\to$ Thm $\Longrightarrow$ esd of $\hat{W} = \frac{1}{\sqrt{d}} W \to \mu_{sc}$.

μ_d

Thm: Suppose $p = p(d) \in (0,1)$. (like in Thm)

(1) If $p(d) \cdot d \to \infty$, then $\mu_d \to \mu_{sc}$.

(2) If $p(d) \cdot d \to c \in (0, \infty)$ (e.g. if $p = \frac{c}{d}$)

then $\mu_d \to \nu_c \neq \mu_{sc}$.

(Idea: column i of $W \equiv 0 \Rightarrow We_i = 0 \Rightarrow 0$ eigval.

$\mathbb{P}[\text{column} = 0] = (1-p)^d = \left(1 - \frac{c}{d}\right)^d \to e^{-c}$ const.)

↳ eigval dist ≈  ← e^{-c} fraction.

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② Extreme eigenvalues:

Note $\mu_d \xrightarrow{(W,P)} \mu_{sc}$ DOES NOT mean $\lambda_1\left(\frac{1}{\sqrt{d}}W\right) \to 2$ (the $\frac{1}{\sqrt{d}}W$ is $\hat{W}$)

For f bdd cts., $\frac{1}{d}\sum_{i=1}^{d} f(\lambda_i(\hat{W}))$ receives only contribution of $\frac{1}{d}\cdot f(\lambda_1) = O\left(\frac{1}{d}\right) \to 0$.

Expect $\begin{cases} \lambda_1(\hat{W}) \xrightarrow{(P)} 2 \\ \|\hat{W}\| \xrightarrow{(P)} 2 \end{cases}$, but not immediate from Thm.

More delicate trace/moment method to control these:

E.g. to control norm, $\lambda_i\left(\frac{1}{\sqrt{d}}W\right)^{2k}$ for $i=1$ or d.

$$P\left\{\left\|\frac{1}{\sqrt{d}}W\right\| \geq t\right\} = P\left\{\left\|\frac{1}{\sqrt{d}}W\right\|^{2k} \geq t^{2k}\right\}$$

$$\leq P\left\{\mathrm{Tr}\left(\left(\frac{1}{\sqrt{d}}W\right)^{2k}\right) \geq t^{2k}\right\}$$

$$\overset{(\text{Markov})}{\leq} \frac{\mathbb{E}\,\mathrm{Tr}\left(\left(\frac{1}{\sqrt{d}}W\right)^{2k}\right)}{t^{2k}}$$

CT $\overset{(\text{Lem})}{=} \dfrac{C_k \cdot d \cdot (1 + o(1))}{t^{2k}}$

$\overset{(C_k \le 2^{2k})}{\le} \dfrac{2^{2k} \cdot d \cdot (1+o(1))}{t^{2k}} = (1+o(1)) \cdot \left(\dfrac{2}{t}\right)^{2k} \cdot d$

To have this $\to 0$ as $d \to \infty$, need

$t \ge 2\, d^{1/2k+\delta}$

"$t(d)$" $\rightsquigarrow$ $\mathbb{P} \le (1+o(1)) \left(\dfrac{2}{2 \cdot d^{1/2k+\delta}}\right)^{2k} \cdot d$

$= O(d^{-2k\delta}) \to 0.$

$\Longrightarrow$ $\|\hat{w}^{(d)}\| \le O(d^{\varepsilon})$ for any $\varepsilon > 0$ whp.

But want/expect: $\|\hat{w}^{(d)}\| \le 2 + \varepsilon$ for any $\varepsilon > 0$ whp.

Next time: more precise bounds of estimates on $\mathbb{E}\,\mathrm{Tr}\,\hat{W}^{2k} \Longrightarrow$ better bounds on $\|W\|$