Statistical inference of a ranked community in a directed graph

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INFORMS APS: Finding Hidden Structure in Networks
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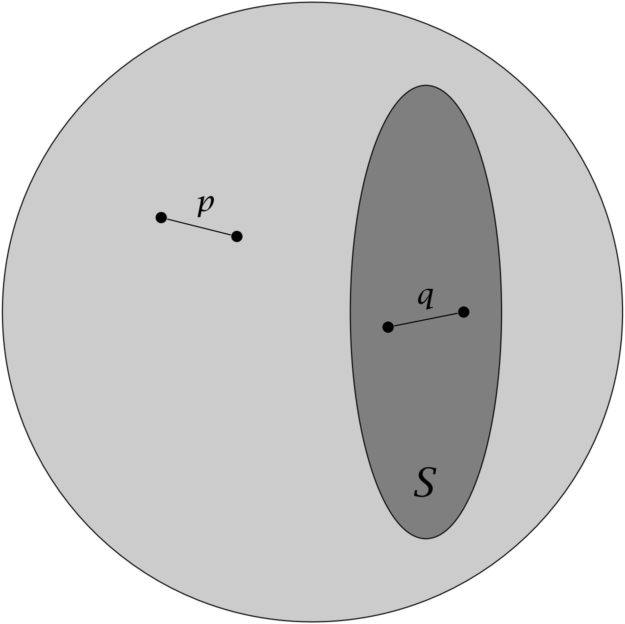
Statistical questions:

- **Detection:** Test $G \sim \mathcal{G}(n, p, q, 0)$ vs. $G \sim \mathcal{G}(n, p, q, k)$?
- **Recovery:** When $G \sim \mathcal{G}(n, p, q, k)$, recover S?


p
q
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(Strong) detection: find **tests** $t_n : \{\text{graphs}\} \to \{0,1\}$ with

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or **separating statistics** $f_n : \{\text{graphs}\} \to \mathbb{R}$ with

$$\mathop{\mathbb{E}}_{G \sim \mathcal{G}(n,p,q,k)}[f_n(G)] - \mathop{\mathbb{E}}_{G \sim \mathcal{G}(n,p,q,0)}[f_n(G)] \gg \sqrt{\mathop{\mathrm{Var}}_{G \sim \mathcal{G}(n,p,q,k)}[f_n(G)]} + \sqrt{\mathop{\mathrm{Var}}_{G \sim \mathcal{G}(n,p,q,0)}[f_n(G)]}.$$

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Recovery: find **estimators** $\widehat{S} : \{\text{graphs}\} \to \binom{[n]}{k}$ with

$$\mathop{\mathbb{E}}_{(S,G) \sim \mathcal{G}(n,p,q,k)} \frac{|S \triangle \widehat{S}(G)|}{k} = \begin{cases} 1 - \Omega(1) & \text{(weak)} \\ o(1) & \text{(strong)} \end{cases}$$

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- **Statistical-computational gap:** For some parameter choices, detection and/or recovery are **possible** but (conjecturally) **computationally hard**.

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- **Detection-recovery gap:** For some parameter choices, detection is easy while recovery is hard or impossible.

  For example, sometimes can detect subgraph merely by **counting total edges**, but this alone does not help to recover S.

[Bhaskara et al ’10; Hajek, Wu, Xu ’15; Verzelen, Arias-Castro ’15; Ma, Wu ’16; Chen, Xu ’16; Brennan, Bresler, Huleihel ’18; Schramm, Wein ’22; Dhawan, Mao, Wein ’23]

Question: What is the analog of this story over **directed graphs**?

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$$\left\{ \begin{array}{ll} i \longrightarrow j & \text{with probability } \left(\frac{1}{2} + q\right) p \\ i \longleftarrow j & \text{with probability } \left(\frac{1}{2} - q\right) p \\ \text{no edge} & \text{with probability } 1 - p \end{array} \right\};$$

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otherwise, set

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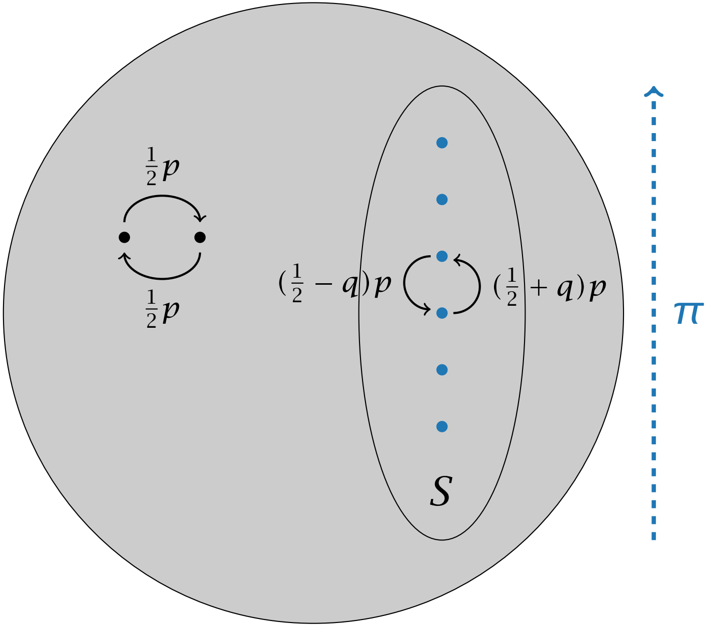
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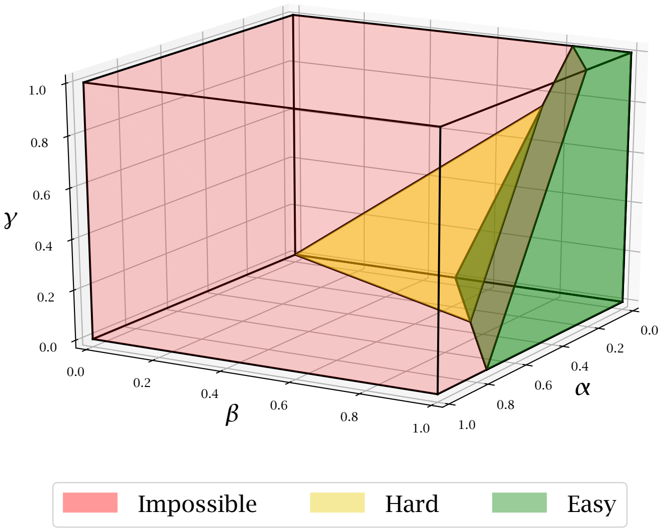
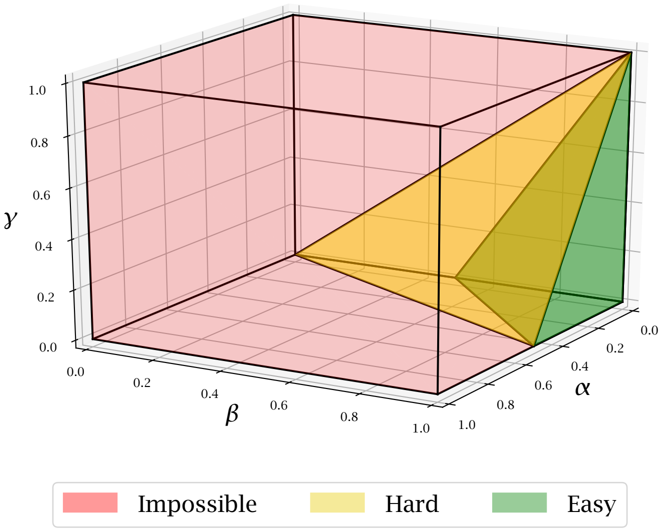
- **Noisy sorting:** Closer to our model—simultaneously observe all pairwise comparisons ($k = n$ and $p = 1$) once, each correct with probability $\frac{1}{2} + q$.

Key difference: possibility of **community structure**, i.e., of having $k < n$.

Main Results: Detection



Main Results: Recovery



The same main phenomena still happen (indeed are a little simpler!) when the community structure is carried entirely by the **edge direction** rather than edge density.

Algebraic Encoding

Standard perspective for graph setting: encode G into $\{0, 1\}$-valued adjacency matrix $\boldsymbol{Y} = \boldsymbol{Y}^\top$,

$$Y_{ij} := \left\{ \begin{array}{rl} +1 & \text{if } i \sim j \\ 0 & \text{if } i \not\sim j \text{ or } i = j \end{array} \right\}.$$

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For oriented directed graph G, natural to encode into $\{0, +1, -1\}$-valued $\boldsymbol{Y} = -\boldsymbol{Y}^\top$,

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Problems may then be viewed over $\{0, +1, -1\}^{n \times n}_{\mathsf{antisym}}$.

Lower Bound Idea (Briefly)

Form of hardness results: no sequence of polynomials of **low degree** $\lesssim (\log n)^{1+\varepsilon}$ can achieve detection (separation) or recovery (of indicator of support S).

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Mild complications: orthogonal polynomials must be taken over null model, "oriented directed Erdős-Rényi graph," whose (adjacency matrix) entries are **sparse Rademacher**:

$$\left\{ \begin{array}{rl} +1 & \text{with probability } \frac{1}{2}p \\ -1 & \text{with probability } \frac{1}{2}p \\ 0 & \text{with probability } 1-p \end{array} \right\}.$$

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$\widetilde{\boldsymbol{Y}}$ is **Hermitian** and behaves like $\mathbb{C}$-valued spiked matrix model. More complicated "signal part," looking like:

$$\mathbb{E}[\widetilde{\boldsymbol{Y}} \mid S, \pi] = \begin{bmatrix} 0 & +i & \cdots & +i \\ -i & 0 & \cdots & +i \\ \vdots & \vdots & \ddots & \vdots \\ -i & -i & \cdots & 0 \end{bmatrix}, \quad \text{rank} = k \sim n^\beta$$

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For recovery, when $k = n$, then **ranking by wins** (i.e., by the W_i scores) achieves the optimal threshold.

Future directions: What does your favorite graph problem (cut, coloring, clique, independent set) or model (stochastic block model, random geometric graph) look like over directed graphs?

Thank you!