Assignment 1

Probability Theory II

(EN.553.721, Spring 2026)

Assigned: January 30, 2026   Due: 11:59pm EST, February 11, 2026

Solve all three problems. Each problem is worth an equal amount towards your grade.

Submit solutions in LaTeX. Write in complete sentences. Include and justify all steps of your arguments, but avoid writing excessive explanation that is not contributing to your solution.

See my website for policies about late submissions, collaboration, and use of AI assistants.

Problem 1 (Convergence and limit theorems)**.** This problem clarifies some details surrounding the limit theorems you have seen in Probability Theory I and at the beginning of this class.

1. Let $X_1, X_2, \dots \in \mathbb{R}$ be random variables and $c \in \mathbb{R}$ a constant. Show that $X_n \to c$ in probability if and only if $X_n \Rightarrow c$ in distribution.

2. Construct $X_1, X_2, \dots \in \mathbb{R}$ random variables such that $X_n \to 0$ in probability but not almost surely.

3. Suppose $X_1, X_2, \dots$ are i.i.d. satisfying $\mathbb{E}X_i = 0$ and $\mathbb{E}X_i^2 = 1$. Show that there does not exist a random variable Z such that $\widehat{S}_n := \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i \to Z$ in probability. Explain why this does not contradict the CLT.

   (Hint: Show that, if this convergence did happen, then $\widehat{S}_{2n} - \widehat{S}_n \to 0$ in probability. Derive a contradiction by showing that this expression must instead converge in distribution to a (non-trivial) Gaussian random variable.)

4. Show that there exists $c > 0$ such that the following holds for arbitrarily large n. Let $X_1, \dots, X_n \sim \mathsf{Unif}(\{\pm 1\})$ and $N \sim \mathcal{N}(0, 1)$. Then,

   $$\left| \mathbb{P}\left[\frac{1}{\sqrt{n}} \sum_{i=1}^n X_i \leq 0 \right] - \mathbb{P}[N \leq 0] \right| \geq \frac{c}{\sqrt{n}}.$$

   This shows that the error bound of the Berry-Esséen theorem is tight in general.

Problem 2 (Stirling's asymptotic)**.** In this problem, you will show that the central limit theorem (CLT) implies Stirling's approximation for the factorial (a purely deterministic statement!). Recall that $\mathsf{Exp}(\lambda)$ is the *exponential* measure, with density $\mathbb{1}\{x \geq 0\}\lambda \exp(-\lambda x)$. Let $X_1, \dots, X_n \sim \mathsf{Exp}(1)$ be i.i.d. and let $S_n := \sum_{i=1}^n X_i$.

1. Show that $\mathbb{E}X_i = \mathsf{Var}\, X_i = 1$, and thus $\mathbb{E}S_n = \mathsf{Var}\, S_n = n$.

2. Show that, for each $n \geq 1$, S_n has density
$$\rho_n(x) = \mathbb{1}\{x \geq 0\} \frac{1}{(n-1)!} x^{n-1} \exp(-x).$$
(**Hint:** Use induction to compute the integrals involved.)

3. Show that
$$\mathbb{E}\left|\frac{S_n - n}{\sqrt{n}}\right| = \frac{2\exp(-n)n^{n+1/2}}{n!}.$$
(**Hint:** Use the density of S_n from Part 2 to write the expectation as an integral. Split the integral into two parts to handle the absolute value appearing. Then, use integration by parts. You should be able to avoid computing any complicated integrals.)

4. Use the CLT applied to the X_i to conclude that
$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi}\exp(-n)n^{n+1/2}} = 1,$$
by applying the CLT to the test function $f(x) = |x|$. Note that, since this function is unbounded, you will have to approximate it by bounded continuous functions.

Problem 3 (Extreme value theory)**.** This problem will introduce a different but equally important class of limit theorem beyond those you have seen in Probability Theory I. Let $X_1, X_2, \dots$ be i.i.d. random variables (with law specified in the parts below) and write $M_n := \max_{i=1}^n X_i$. As a general hint, in all cases, look for a formula for $\mathbb{P}[M_n \leq t]$.

1. Suppose $X_i \sim \mathsf{Unif}([0,1])$. Show that $n(1 - M_n) \Rightarrow M$, where $\mathsf{Law}(M) = \mathsf{Exp}(1)$.

2. Suppose $X_i \sim \mathsf{Exp}(1)$. Show that $(M_n - \log n) \Rightarrow M$, where M is a random variable with density function $\exp(-x - \exp(-x))$ (the nested exponential is not a typo, and the density is over all $x \in \mathbb{R}$, not just non-negative x).

3. This part is in a different setting from the above two, preparing for the last part. Show that, if $N \sim \mathcal{N}(0,1)$, then, for any $t \geq 0$,
$$\mathbb{P}[N \geq t] \leq \frac{1}{2}\exp\left(-\frac{t^2}{2}\right).$$
(**Hint:** Write out the integral, change variables to make it an integral from 0 to ∞, and expand the square.)

4. In the original setting, let $X_i \sim \mathcal{N}(0,1)$. Using Part 3, show that, for any $\epsilon > 0$,
$$\lim_{n\to\infty} \mathbb{P}[M_n \leq \sqrt{(2+\epsilon)\log n}] = 1.$$
You do not need to prove it, but know that in fact $M_n / \sqrt{2\log n} \to 1$ in probability.