LECTURE 2: Lindeberg method for CLT.

Thm: (weak CLT) $X_1, X_2, \ldots \overset{\text{iid}}{\sim} \mu$ with

- $\mathbb{E} X_i = c$
- $\operatorname{Var} X_i = \mathbb{E} X_i^2 - c^2 = \sigma^2$

⊛ $\mathbb{E}|X_i|^3 =: R < \infty$ ← can remove w/ truncation.

$S_n := \sum_{i=1}^n (X_i - c)$, then $\operatorname{Law}\left(\frac{1}{\sqrt{n}} S_n\right) \xrightarrow{(w)} \mathcal{N}(0, \sigma^2)$

$\Updownarrow$

$\frac{1}{\sqrt{n}} S_n \Rightarrow N$, $\sim \mathcal{N}(0,\sigma^2)$

i.e. $\boxed{\mathbb{E} f\left(\frac{1}{\sqrt{n}} S_n\right) \xrightarrow[n\uparrow\infty]{} \mathbb{E} f(N)}$ $\forall f: \mathbb{R} \to \mathbb{R}$ $\boxed{\text{bdd. cts.}}$

OR

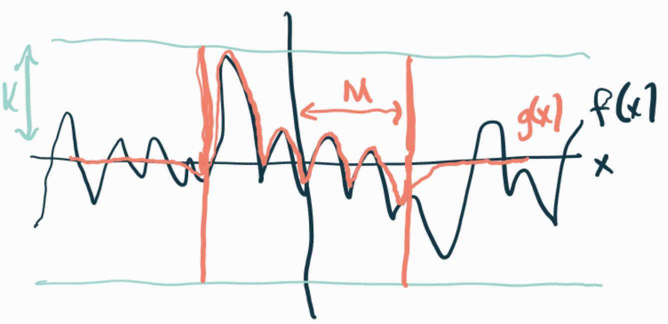
$\mathbb{P}\left\{\frac{1}{\sqrt{n}} S_n \le t\right\} \xrightarrow[n\uparrow\infty]{} \mathbb{P}\{N \le t\}$ $\forall t \in \mathbb{R}$.

Lem: $\mu_1, \mu_2, \ldots, \mu_\infty$ prob. meas. on $\mathbb{R}$. Then $\mu_n \xrightarrow{(w)} \mu_\infty$, (i.e. $\int f d\mu_n \to \int f d\mu_\infty$ $\forall f$ bdd, cts.) iff

$\int f d\mu_n \to \int f d\mu_\infty$ $\forall f$ smooth, compactly supported.

Pf: $\Rightarrow$: immediate. $\Leftarrow$: let $f: \mathbb{R} \to \mathbb{R}$ bdd cts. ($|f(x)| \le K$).

Fix $\varepsilon > 0$, $M > 0$.



Claim: $\exists\, g: \mathbb{R} \to \mathbb{R}$ smooth, cpt. supp. such that:

- $|f(x) - g(x)| \le \varepsilon$ $\forall x \in [-M, M]$
- $|g(x)| \le 2K$ $\forall x \in \mathbb{R}$.

$$\left|\int f d\mu_n - \int f d\mu_\infty\right| \le \underbrace{\left|\int f d\mu_n - \int g d\mu_n\right|}_{①} + \overbrace{\underbrace{\left|\int g d\mu_n - \int g d\mu_\infty\right|}_{②}}^{\text{ass'n} \Rightarrow \text{term} \to 0} + \underbrace{\left|\int f d\mu_\infty - \int g d\mu_\infty\right|}_{③}$$

$$\left|\int f\,d\mu_n - \int g\,d\mu_n\right| \le \int \underbrace{|f-g|}_{\le 3K}\,d\mu_n \le \varepsilon + 3K\cdot \mu_n(\mathbb{R}\setminus[-M,M])$$

Claim: term $\to \mu_\infty(\mathbb{R}\setminus[-M,M])$

$$\left|\int f\,d\mu_\infty - \int g\,d\mu_\infty\right| \le \cdots \le \varepsilon + 3K\mu_\infty(\mathbb{R}\setminus[-M,M])$$

$$\Rightarrow \limsup_{n\uparrow\infty} \left|\int f\,d\mu_n - \int f\,d\mu_\infty\right| \le 2\varepsilon + 6K\mu_\infty(\mathbb{R}\setminus[-M,M])$$

Then, take $\varepsilon \to 0$, $M \to \infty$. ∎

Rk: • Lem: suffices to look @ smooth, cpt. supp. test f.
• Lévy thm: suffices to look @ test $f(x) = \exp(itx)$.
• Sometimes: suffices to look @ polynomial f ("moment method")

Pf of Thm: WLOG: $c = \mathbb{E}X_i = 0$, $\sigma^2 = \mathrm{Var}\,X_i = \mathbb{E}X_i^2 = 1$.

Introduce $Y_1, \dots, Y_n \overset{iid}{\sim} \mathcal{N}(0,1)$ and $N \sim \mathcal{N}(0,1)$.

Rk: $\left\{\begin{matrix} \mathbb{E}X_i = \mathbb{E}Y_i = 0 \\ \mathbb{E}X_i^2 = \mathbb{E}Y_i^2 = 1 \end{matrix}\right\}$.

Want to show: $\forall f$ smooth, cpt. supp,

Rk: $\mathrm{Law}(N) = \mathrm{Law}\left(\frac{Y_1+\dots+Y_n}{\sqrt{n}}\right)$

$$\begin{matrix} \mathbb{E}f\left(\frac{1}{\sqrt{n}}S_n\right) & \xrightarrow[n\uparrow\infty]{} & \mathbb{E}f(N) \\ \| & & \| \\ \mathbb{E}f\left(\frac{X_1+\dots+X_n}{\sqrt{n}}\right) & & \mathbb{E}f\left(\frac{Y_1+\dots+Y_n}{\sqrt{n}}\right) \end{matrix}$$

(Prop: (Basic Gaussian properties) $A \sim \mathcal{N}(\mu,\sigma^2)$, $A' \sim \mathcal{N}(\mu',\sigma'^2)$ indep.
• $\mathrm{Law}(\alpha A + \beta) = \mathcal{N}(\alpha\mu+\beta, \alpha^2\sigma^2)$ → gives pf. of Rk.
• $\mathrm{Law}(A + A') = \mathcal{N}(\mu+\mu', \sigma^2+\sigma'^2)$.)

Simple idea: Replace X_i one by one by Y_i and show $\mathbb{E}f(\text{—})$ stays roughly same.

$$\Delta^{(n)} := \mathbb{E} f\left(\frac{X_1+\dots+X_n}{\sqrt{n}}\right) - \mathbb{E} f\left(\frac{Y_1+\dots+Y_n}{\sqrt{n}}\right)$$

$$= \sum_{k=1}^{n} \underbrace{\left[\mathbb{E} f\left(\frac{Y_1+\dots+Y_{k-1}+X_k+X_{k+1}+\dots+X_n}{\sqrt{n}}\right) - \mathbb{E} f\left(\frac{Y_1+\dots+Y_{k-1}+Y_k+X_{k+1}+\dots+X_n}{\sqrt{n}}\right)\right]}_{=:\ \Delta_k^{(n)}}$$

(all but k^{th} term)

<u>Obs 1:</u> $$\Delta_k^{(n)} = \mathbb{E} f\Big(\underset{O(1)}{\underset{\parallel}{Z_k}} + \underset{O(\frac{1}{\sqrt{n}})}{\underset{\parallel}{\frac{X_k}{\sqrt{n}}}}\Big) - \mathbb{E} f\Big(\underset{O(1)}{\underset{\parallel}{Z_k}} + \underset{O(\frac{1}{\sqrt{n}})}{\underset{\parallel}{\frac{Y_k}{\sqrt{n}}}}\Big)$$

Since f smooth, can expand by Taylor:

Taylor remainder

$$f\left(Z_k + \frac{X_k}{\sqrt{n}}\right) = f(Z_k) + f'(Z_k)\frac{X_k}{\sqrt{n}} + \frac{1}{2} f''(Z_k)\frac{X_k^2}{n} + \boxed{\frac{1}{6} f'''(\tilde{Z})\frac{X_k^3}{n^{3/2}}}$$

$$\tilde{Z} = \tilde{Z}\left(Z_k, \frac{X_k}{\sqrt{n}}\right)$$

<u>Obs 2:</u> Z_k ind. of both X_k and Y_k (!) $\Longrightarrow$

$$\left|\mathbb{E} f\left(Z_k + \frac{X_k}{\sqrt{n}}\right) - \left[\mathbb{E} f(Z_k) + \mathbb{E} f'(Z_k)\frac{\overset{\text{ind.}}{\boxed{\mathbb{E} X_k}}}{\sqrt{n}} + \frac{1}{2}\mathbb{E} f''(Z_k)\frac{\overset{\text{ind.}}{\boxed{\mathbb{E} X_k^2}}}{n}\right]\right| \le \frac{1}{6}\|f'''\|_{L^\infty}\frac{\overset{R}{\mathbb{E}|X_k|^3}}{n^{3/2}}$$

$$\left|\mathbb{E} f\left(Z_k + \frac{Y_k}{\sqrt{n}}\right) - \left[\mathbb{E} f(Z_k) + \mathbb{E} f'(Z_k)\frac{\boxed{\mathbb{E} Y_k}}{\sqrt{n}} + \frac{1}{2}\mathbb{E} f''(Z_k)\frac{\boxed{\mathbb{E} Y_k^2}}{n}\right]\right| \le \frac{1}{6}\|f'''\|_{L^\infty}\frac{\mathbb{E}|Y_k|^3}{n^{3/2}}$$

$\|f'''\|_{L^\infty} = \sup_{x\in\mathbb{R}} |f'''(x)| < \infty$ since f smooth + cpt. supp.

$$|\Delta_k^{(n)}| = \left|\mathbb{E} f\left(Z_k + \frac{X_k}{\sqrt{n}}\right) - \mathbb{E} f\left(Z_k + \frac{Y_k}{\sqrt{n}}\right)\right|$$

$$\le \frac{1}{6}\|f'''\|_{L^\infty}\left(\overset{R}{\mathbb{E}|X_k|^3} + \mathbb{E}|Y_k|^3\right)\frac{1}{n^{3/2}} = C(f, \overset{\text{Law}(X_i)}{\mu}) \cdot \frac{1}{n^{3/2}}.$$

$$|\Delta^{(n)}| \le \sum_{k=1}^{n} |\Delta_k^{(n)}| \le C(f,\mu)\frac{1}{\sqrt{n}}, \quad \to 0 \text{ as } n\to\infty. \quad \blacksquare$$

$$|\Delta^{(n)}| = \left|\mathbb{E} f\left(\tfrac{1}{\sqrt{n}} S_n\right) - \mathbb{E} f(N)\right|.$$

Remarks

① Even for Y_i with $\mathbb{E}Y_i = \overset{0}{\overset{\shortparallel}{\mathbb{E}X_i}}$, $\mathbb{E}Y_i^2 = \mathbb{E}X_i^2$, $\mathbb{E}|Y_i|^3 < \infty$, get

$$\lim_{n\to\infty}\left|\mathbb{E}f\left(\frac{X_1+\dots+X_n}{\sqrt{n}}\right) - \mathbb{E}f\left(\frac{Y_1+\dots+Y_n}{\sqrt{n}}\right)\right| = 0.$$

Lindeberg can imply universality of a limit without knowing what limit is.

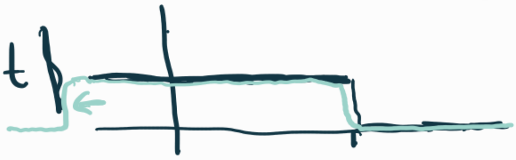
② Immediately gives non-asymptotic (depending on n), quantitative bounds:

$$\left|\mathbb{E}f\left(\tfrac{1}{\sqrt{n}}S_n\right) - \mathbb{E}f(N)\right| \le \left(\frac{1}{3} + \frac{1}{6}\mathbb{E}|X_1|^3\right)\|f'''\|_{C^0}\frac{1}{\sqrt{n}}.$$

Faster convergence for (1) "flatter" f, (2) lighter-tailed X_i.

Thm: (Berry–Esséen) $\forall t \ge 0$,

$$\left|\mathbb{P}\left\{\tfrac{1}{\sqrt{n}}S_n \le t\right\} - \mathbb{P}\{N \le t\}\right| \le \left(1 + \mathbb{E}|X_1|^3\right)\frac{1}{\sqrt{n}}$$ (comparison of cdf's)

Pf idea: $f(x) \approx \mathbb{1}\{x \le t\}$  , then exact same ideas.
smoothed version of step fn.

Rk: Source of $O\big(\overbrace{\tfrac{1}{\sqrt{n}}}^{= n\cdot\frac{1}{n^{3/2}}}\big)$ error: from remainder in Taylor exp.

If $\mathbb{E}X_i^k = \mathbb{E}N^k = \mathbb{E}Y_i^k$ $\forall\, 1 \le k \le \ell$, $\mathbb{E}|X_i|^{\ell+1} < \infty$,
(previously $\ell = 2$)

same argument w/ higher order Taylor (order ℓ) gives error of

$$O\Big(n\cdot\underbrace{\left(\tfrac{1}{\sqrt{n}}\right)^{\ell+1}}_{=\frac{1}{n^{\frac{\ell+1}{2}}}}\Big) = O\left(\frac{1}{n^{\frac{\ell-1}{2}}}\right).$$

Ex: Say X_i symmetric $(\mathrm{Law}(X) = \mathrm{Law}(-X)) \Rightarrow \mathbb{E}X_i^3 = 0 = \mathbb{E}N^3$
$\rightsquigarrow \ell = 3$ CLT $\to O(1/n)$ error bound.