LECTURE 25: Last time, saw motivation for stochastic integrals: given processes $(H(t))$, $(W(t))$ adapted to $(\mathcal{F}(t))$, build

$$I(t) = \int_0^t H(s)\,dW(s) = \lim \sum_{i=0}^{n-1} H(t_i)\big(W(t_{i+1}) - W(t_i)\big)$$

Ex: $H(t) = W(t) = B(t)$, Brownian motion. Look at $I(1)$:

$$S_n := \sum_{j=0}^{n-1} B\left(\tfrac{j}{n}\right)\left(B\left(\tfrac{j+1}{n}\right) - B\left(\tfrac{j}{n}\right)\right) \quad ① \qquad \left(t_j := \tfrac{j}{n}\right)$$

$$= \sum B\left(\tfrac{j+1}{n}\right)\left(B\left(\tfrac{j+1}{n}\right) - B\left(\tfrac{j}{n}\right)\right) - \sum \left(B\left(\tfrac{j+1}{n}\right) - B\left(\tfrac{j}{n}\right)\right)^2 \quad ②$$

$$2S_n = ① + ② = \underbrace{\sum \left(B\left(\tfrac{j+1}{n}\right)^2 - B\left(\tfrac{j}{n}\right)^2\right)}_{\substack{\text{telescopes into} \\ B(1)^2 - B(0)^2 = B(1)^2}} - \underbrace{\sum \left(B\left(\tfrac{j+1}{n}\right) - B\left(\tfrac{j}{n}\right)\right)^2}_{\xrightarrow{(a.s.)} 1 \ \text{(HW 5)}}$$

$$\boxed{I(1) = \tfrac{1}{2}\left(B(1)^2 - 1\right)}. \quad \text{Similarly } \boxed{I(t) = \tfrac{1}{2}\left(B(t)^2 - t\right)}.$$

$\neq$

Rk 1: Result is a martingale.

Rk 2: If B were C^1, would have

$$\int_0^1 B(s)\,dB(s) = \int_0^1 B(s)B'(s)\,ds = \frac{B(s)^2}{2}\Big|_0^1 = \boxed{\frac{B(1)^2}{2}}.$$

$\rightsquigarrow$ roughness of BM paths makes non-trivial change!

Rk 3: Consider variant $\tilde{S}_n = \sum_{j=0}^{n-1} B\left(\tfrac{j+1}{n}\right)\left(B\left(\tfrac{j+1}{n}\right) - B\left(\tfrac{j}{n}\right)\right)$

$\rightsquigarrow \tilde{I}(t) = \frac{1}{2}\left(B(t)^2 + t\right)$. $\quad \tilde{\tilde{S}}_n = \sum_{j=0}^{n-1} \left(\frac{B\left(\frac{j}{n}\right) + B\left(\frac{j+1}{n}\right)}{2}\right)\left(B\left(\tfrac{j+1}{n}\right) - B\left(\tfrac{j}{n}\right)\right)$

$\rightsquigarrow \tilde{\tilde{I}}(t) = \frac{1}{2}B(t)^2$, = "classical" answer.

$I(t)$ = Itô integral , $\tilde{I}(t)$ = Stratonovich integral.

Thm: Suppose $H(t)$ is adapted to $(\mathcal{F}^+(t))$ and (a.s. pathwise) left- or right-continuous, with $E\left[\int_0^t H(s)^2\,ds\right] < \infty$ $\forall t \geq 0$. Then, for any $t > 0$ and $0 = t_0^{(n)} < \cdots < t = t_n^{(n)}$ with $\max_i |t_{i+1}^{(n)} - t_i^{(n)}| \to 0$,

$$\sum_{j=0}^{n-1} H(t_j^{(n)})\left(B(t_{j+1}^{(n)}) - B(t_j^{(n)})\right) \xrightarrow{(L^2)} I(t).$$

Further, $I(t)$ has a continuous version and is a martingale w/ $I(t) \in L^2$. This is the Itô integral, denoted

$$I(t) = \int_0^t H(s)\,dB(s).$$

Def: Suppose $\tilde{b}(t)$, $\tilde{\sigma}(t)$ adapted and left/right cts. having $E\left[\int_0^t (\tilde{\sigma}(s)^2 + |\tilde{b}(s)|)\,ds\right] < \infty$ $\forall t \geq 0$. Let

$$X(t) := \int_0^t \tilde{\sigma}(s)\,dB(s) + \int_0^t \tilde{b}(s)\,ds$$

similarly for $\sigma(t, X(t))$, $b(t, X(t))$.

We call X an Itô process or generalized (Itô) diffusion. Suppose further that $\tilde{\sigma}(t) = \sigma(X(t))$ and $\tilde{b}(t) = b(X(t))$. Then we say X is an Itô diffusion that solves the SDE

$$dX(t) = \sigma(X(t))\,dB(t) + b(X(t))\,dt \longrightarrow$$

Rk: Separate non-trivial Theorem that $X(t)$ actually gives solution.

- First term $\int_0^t \tilde{\sigma}(s)\,dB(s)$ is martingale part.
- Second term $\int_0^t \tilde{b}(s)\,ds$ is drift part: if $\tilde{b}(t)$ cts., is differentiable! In general, bounded variation — much nicer than BM.

Thm: Itô diffusion is Markov process (w/ nice σ and b).

Itô formula: Idea — function of Itô process is another Itô process.

Deterministic case: $dX(t) = b(X(t))\,dt \leftrightarrow X'(t) = b(X(t))$, $\quad X'(t) = \frac{d}{dt}X(t)$.

$$\to \frac{d}{dt} f(X(t)) = f'(X(t))\, X'(t)$$
$$= f'(X(t))\, b(X(t))$$

$$\leftrightarrow df(X(t)) = f'(X(t))\, b(X(t))\, dt.$$

Discretized / infinitesimal viewpoint:

$$X(t+\delta) = X(t) + b(X(t))\cdot\delta + o(\delta)$$
$$f(X(t+\delta)) = f\big(X(t) + b(X(t))\cdot\delta + o(\delta)\big)$$
$$= f(X(t)) + f'(X(t))\cdot b(X(t))\cdot\delta + o(\delta)$$

Stochastic case:

$$X(t+\delta) = X(t) + b(X(t))\cdot\delta + \sigma(X(t))\underbrace{\big(B(t+\delta) - B(t)\big)}_{\boxed{\asymp\sqrt{\delta}}} + o(\delta)$$

$$f(X(t+\delta)) = f(\;\cdot\;\text{——}\;)$$
$$= f(X(t)) + f'(X(t))\Big[b(X(t))\cdot\delta + \sigma(X(t))\big(B(t+\delta) - B(t)\big) + o(\delta)\Big]$$
$$+ \frac{f''(X(t))}{2}\Big[b(X(t))\cdot\delta + \sigma(X(t))\big(B(t+\delta) - B(t)\big) + o(\delta)\Big]^2$$
$$+ O(\delta^{3/2})$$
$$= f(X(t)) + f'(X(t))\, b(X(t))\cdot\delta + f'(X(t))\,\sigma(X(t))\cdot\big(B(t+\delta) - B(t)\big)$$
$$+ \frac{f''(X(t))}{2}\,\sigma(X(t))^2\cdot(\delta + \text{fluctuations})$$
$$\approx f(X(t)) + \left(f'(X(t))\, b(X(t)) + \frac{f''(X(t))}{2}\,\sigma(X(t))^2\right)\delta + f'(X(t))\,\sigma(X(t))\big(B(t+\delta) - B(t)\big).$$

Thm: (slightly informal) X Itô diffusion w/

$$dX(t) = \sigma(t, X(t))\, dB(t) + b(t, X(t))\, dt$$

and f smooth, then

$$df(X(t)) = f'(X(t))\,\sigma(t, X(t))\, dB(t) + \left(f'(X(t))\, b(t, X(t)) + \frac{f''(X(t))}{2}\sigma(t, X(t))^2\right) dt.$$

Rk: Shortcut: "$(dB(t))^2 = dt$".

Ex: Consider SDE

$$dX(t) = \mu X(t)\, dt + \sigma X(t)\, dB(t)$$

GEOMETRIC BM. Stock price model in Black-Scholes theory.

Claim: solution given by (multiples of) $X(t) = \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B(t)\right)$.

Itô w/ $f(x) = \log x$: suppose $X(t)$ is solution, then

$$d[\log X(t)] = \frac{1}{\cancel{X(t)}} \cdot \sigma \cancel{X(t)}\, dB(t) + \left(\frac{1}{\cancel{X(t)}} \cdot \mu \cancel{X(t)} - \frac{1}{2\cancel{X(t)^2}} \cdot \sigma^2 \cancel{X(t)^2}\right) dt$$

$$= \sigma\, dB(t) + \left(\mu - \frac{\sigma^2}{2}\right) dt$$

Integrating, $\log X(t) = \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B(t)$.

Further topics:

- Markov process theory of diffusions: generator, heat eqn. analog, connections w/ parabolic PDE.
- Qualitative theory: cannot solve most SDE in closed form. How do they behave as $t \to \infty$? How do classical DE change under "random forcing"?
- Pointwise theory: though $\int H(s)\, dB(s)$ doesn't exist for BM as classical integral, can work around with "rough path integral" theory — much more general SDE theory (cf. Lyons, Hairer).