LECTURE 24: Martingales in continuous time.

Def: $X(t)$ is a <u>martingale</u> w.r.t $(\mathcal{F}(t))$ if

(0) adapted

(1) $\mathbb{E}|X(t)| < \infty \quad \forall t$

(2) $\mathbb{E}[X(t) \mid \mathcal{F}(s)] = X(s)$ a.s. $\forall s \leq t$.

$\geq$: submgl

$\leq$: supermgl

Prop: BM is a martingale. (w.r.t $(\mathcal{F}^+(t))$).

Pf: $$\mathbb{E}[B(t) \mid \mathcal{F}^+(s)] = \underbrace{\mathbb{E}[B(s) \mid \mathcal{F}^+(s)]}_{= B(s).} + \underbrace{\mathbb{E}[B(t) - B(s) \mid \mathcal{F}^+(s)]}_{= \mathbb{E}[B(t) - B(s)] = 0.}$$

<u>Parallels to discrete theory:</u>

- Optional stopping
- Maximal inequalities
- Convergence theorems

$\rightarrow$ **Rk:** Often prove by discretization and using discrete results.

- <u>Harmonic functions</u> (HW4): Saw if (Y_n) MC in countable space S, $f: S \to \mathbb{R}$ s.t. $\mathbb{E}_y f(Y_1) = f(y)$, then $(f(Y_n))$ martingale.

$(Tf)(y) := \mathbb{E}_y f(Y_1)$ is discrete transition operator

$Tf = f \implies (f(Y_n))$ martingale.

Cts time: analogous condition $T_t f = f \ \forall t \iff Lf = \frac{d}{dt} T_t f \big|_{t=0} = 0.$

BM in $\mathbb{R}$: $L = \frac{1}{2}\partial^2 \rightsquigarrow f''(x) = 0 \rightarrow$ linear polynomials.

BM in $\mathbb{R}^d$: $L = \frac{1}{2}\Delta \rightarrow \Delta f = 0 \rightsquigarrow$ <u>harmonic functions</u>.

Thm: (Liouville) f harmonic, bounded $\Longrightarrow$ f constant.

Pf: $f(B(t))$ bounded martingale $\Longrightarrow$ converges a.s. to some r.v. F.

bounded in any L^p $\Longrightarrow$ converges in L^p $\Longrightarrow$ converges in L^1.

Note $f(y) = (T_1 f)(y) = \underset{g\sim N(0,I)}{\mathbb{E}} f(y+g) = \underset{y}{\mathbb{E}} f(B(1))$

$$\Longrightarrow \underset{B}{\mathbb{E}} \left| f(B(n+1)) - f(B(n)) \right|$$
$$= \underset{B}{\mathbb{E}} \left| \underset{g}{\mathbb{E}} f(B(n+1)+g) - \underset{g}{\mathbb{E}} f(B(n)+g) \right|$$
$$= \underset{B}{\mathbb{E}} \left| \underset{g}{\mathbb{E}} \left[f(B(n+1)+g) - f(B(n)+g) \right] \right|$$
$$\leq \underset{B,g}{\mathbb{E}} \left| f(B(n+1)+g) - f(B(n)+g) \right|$$
$$= \underset{B}{\mathbb{E}} \left| f(B(n+2)) - f(B(n+1)) \right|.$$

$$\Longrightarrow \mathbb{E} \left| f(B(1)) - f(\overset{0}{\overset{\shortparallel}{B(0)}}) \right| \leq \mathbb{E} \left| f(B(n+1)) - f(B(n)) \right|$$
$$\leq \mathbb{E} \left| f(B(n+1)) - F \right| + \mathbb{E} \left| f(B(n)) - F \right| \to 0.$$

$$\Longrightarrow \underset{g}{\mathbb{E}} \left| f(g) - f(0) \right| = 0 \Longrightarrow f(x) = f(0) \ \forall x. \quad \blacksquare$$

Stochastic differential equation: limit as $\delta \downarrow 0$ of system

"driving" process, e.g. $B(t)$.

$$X(t+\delta) = X(t) + b(X(t),t)\,\delta + \sigma(X(t),t)\left[W(t+\delta) - W(t)\right]$$

$\sigma = 0 \rightsquigarrow X'(t) = b(X(t),t)$ (general ODE) $\rightsquigarrow X(t) - X(0) = \int_0^t b(X(s),s)\,ds.$

$b = 0 \rightsquigarrow X(t) - X(0) = \int_0^t \sigma(X(s),s)\,dW(s)$, like Stieltjes integral.

Naturally related idea: stochastic integral,

$$\int_0^t H(s)\, dW(s) = \lim_{\max|t_i - t_{i-1}| \to 0} \sum_{i=0}^{n-1} H(t_i)\big(W(t_{i+1}) - W(t_i)\big).$$

Q: If H, W stochastic processes, when does this exist? In what sense of limit? How to compute?

Ex: Consider $H = W = B$, BM. $\int_0^1 B(s)\, dB(s) = ?$

$$S_n = \sum_{j=0}^{n-1} B\left(\tfrac{j}{n}\right)\left[B\left(\tfrac{j+1}{n}\right) - B\left(\tfrac{j}{n}\right)\right] = \sum B\left(\tfrac{j+1}{n}\right)\left[B\left(\tfrac{j+1}{n}\right) - B\left(\tfrac{j}{n}\right)\right] - \sum \left(B\left(\tfrac{j+1}{n}\right) - B\left(\tfrac{j}{n}\right)\right)^2$$

$$2S_n = \underbrace{\sum \left[B\left(\tfrac{j+1}{n}\right)^2 - B\left(\tfrac{j}{n}\right)^2\right]}_{B(1)^2 - B(0)^2 = B(1)^2} - \underbrace{\sum \Big(\underbrace{B\left(\tfrac{j+1}{n}\right) - B\left(\tfrac{j}{n}\right)}_{N(0,\frac{1}{n})}\Big)^2}_{\to 1}$$

$$\Longrightarrow \int_0^1 B(s)\, dB(s) = \boxed{\tfrac{1}{2} B(1)^2 - \tfrac{1}{2}}$$

Rk 1: If B were differentiable, would have

$$\int_0^1 B(s)\, dB(s) = \int_0^1 B(s) B'(s)\, ds = \int_0^1 \left(\frac{B(s)^2}{2}\right)' ds$$

$$= \frac{B(1)^2}{2} - \frac{B(0)^2}{2} = \frac{B(1)^2}{2}$$

Non-trivial extra effect of rough random paths of B!

Rk 2: "Other direction" of integral: $\tilde{S}_n = \sum_{j=0}^{n-1} B\left(\tfrac{j+1}{n}\right)\left(B\left(\tfrac{j+1}{n}\right) - B\left(\tfrac{j}{n}\right)\right)$

$\tilde{S}_n^2 \longrightarrow \frac{1}{2} B(1)^2 + \frac{1}{2}$ from same calculations!

$$\tilde{\tilde{S}}_n = \sum_{j=0}^{n-1} \left[\frac{B\left(\frac{j+1}{n}\right) + B\left(\frac{j}{n}\right)}{2}\right]\left[B\left(\frac{j+1}{n}\right) - B\left(\frac{j}{n}\right)\right] \longrightarrow \frac{1}{2} B(1)^2$$

= "classical answer".

$$\left.\begin{array}{l} S_n \longrightarrow \underline{\text{Itô integral}} \\ \tilde{\tilde{S}}_n \longrightarrow \underline{\text{Stratonovich integral}} \end{array}\right\} \text{quantitatively different!}$$

Key idea of Itô choice: integrand H is predictable (behind in time) relative to increments of W (or non-anticipating.)

Interacts nicely w/ stochastic process theory. E.g., if W martingale and H adapted w.r.t. $(\mathcal{F}(t))$, then

$$\int_0^t H(s)\,dW(s) \approx \sum_{j=0}^{n-1} \underbrace{H(t_j)}_{\text{predictable}} \Big[\underbrace{W(t_{j+1}) - W(t_j)}_{\text{martingale increments}}\Big] \quad (0=t_0<\cdots<t_n=1)$$

$= H \cdot W$ (discretized) $\leftarrow$ martingale transform.

Thm: (Informal) If H, W adapted to $(\mathcal{F}(t))$, W martingale, both "nice", then $I(t) := \int_0^t H(s)\,dW(s)$ also martingale.