LECTURE 23: Strong Markov property for Brownian motion.

Def: An r.v. $T \in [0,\infty]$ is a (cts.-time) stopping time w/r/t filtration $(\mathcal{F}(t))_{t\geq 0}$ if $\{T \leq t\} \in \mathcal{F}(t)\ \forall t$.

Prop: If $(\mathcal{F}(t))$ right-cts., equivalent condition is $\{T < t\} \in \mathcal{F}(t)\ \forall t$.

Ex: If $A \subseteq \mathbb{R}$ is open or closed, then hitting time $T := \inf\{t : B(t) \in A\}$ is a stopping time, w/r/t $(\mathcal{F}^+(t))$.

Open case: $\{T < t\} = \{\exists\, s < t : B(s) \in A\}$

$$= \{\exists\, s<t,\ 0<\varepsilon<t-s : B(x) \in A\ \forall x \in (s, s+\varepsilon)\} \quad \text{(continuity of } B(t) + A \text{ open)}$$

$$= \bigcup_{\substack{s<t \\ s\in\mathbb{Q}}} \{B(s) \in A\} \in \mathcal{F}(t) \subset \mathcal{F}^+(t).$$

Exc: closed case – little more complicated!

Last time:

Thm: (Weak Markov) $\forall s \geq 0$, $X(t) := B(s+t) - B(s)$ is a BM indep. of $\mathcal{F}^+(s)$.

Thm: (Strong Markov) Let S be a stopping time (w/r/t $(\mathcal{F}^+(t))$) that is finite a.s. Then, $X(t) := B(S+t) - B(S)$ is a BM indep. of $\mathcal{F}^+(S) := \{A : A \cap \{S \leq t\} \in \mathcal{F}^+(t)\ \forall t \geq 0\}$

Pf: First show for "rounded" times:

$$S_n := \frac{m+1}{2^n} \text{ for } m \text{ s.t. } S \in \left[\frac{m}{2^n}, \frac{m+1}{2^n}\right).$$

Suppose true for these. $X(t)$ clearly a.s. continuous. $X_n(t) := B(S_n + t) - B(S_n)$ $\to$ $X_n(t)$ is BM, $X_n(t) \xrightarrow[n\to\infty]{} X(t)$ a.s. $\forall t$. $\to$ $X(t)$ has BM finite distributions $\Rightarrow$ $X(t)$ is BM. Finally, $\mathcal{F}^+(S) \subseteq \mathcal{F}^+(S_n)$, so $X(t)$ indep of $\mathcal{F}^+(S)$.

$Y_k(t) := B\left(\frac{k}{2^n} + t\right) - B\left(\frac{k}{2^n}\right)$ is BM for each k, indep. of $\mathcal{F}^+\left(\frac{k}{2^n}\right)$

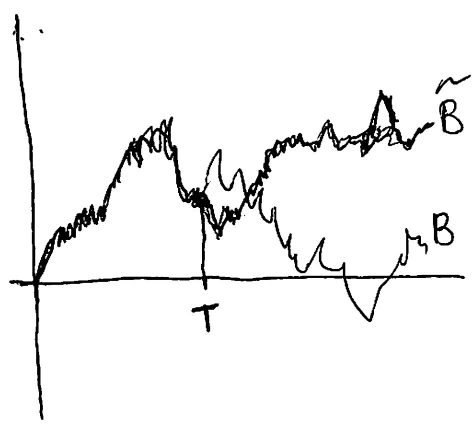
Suppose $E \in \mathcal{F}^+(S_n)$, $A \in \sigma(X_n(t)) \to A = \{X_n \in \tilde{A}\}$ by WMP.

$$
\begin{aligned}
\mathbb{P}[A \cap E] &= \mathbb{P}[\{X_n \in \tilde{A}\} \cap E] \\
&= \sum_{k \geq 0} \mathbb{P}[\{Y_k \in \tilde{A}\} \cap \underbrace{E \cap \{S_n = \tfrac{k}{2^n}\}}_{\in \mathcal{F}^+\left(\frac{k}{2^n}\right)}] \\
&= \sum_{k \geq 0} \underbrace{\mathbb{P}[Y_k \in \tilde{A}]}_{= \mathbb{P}[B \in \tilde{A}], \text{ since all BM.}} \mathbb{P}[E \cap \{S_n = \tfrac{k}{2^n}\}] \\
&= \mathbb{P}[B \in \tilde{A}] \sum_{k \geq 0} \mathbb{P}[E \cap \{S_n = \tfrac{k}{2^n}\}] = \mathbb{P}[B \in \tilde{A}]\,\mathbb{P}[E].
\end{aligned}
$$

$\Longrightarrow X_n$ is BM, indep. of $\mathcal{F}^+(S_n)$. ∎

Cor: (Reflection principle) Let T be stopping time, B be BM. Then,

$$\tilde{B}(t) := \begin{cases} B(t) & \text{if } t \leq T \\ 2B(T) - B(t) & \text{if } t > T \end{cases} \text{ is BM.}$$



Pf: (slightly informal) Let $X(t) = (B(t))_{0 \leq t \leq T}$,

$Y(t) = (B(T+t) - B(t))_{t \geq 0}$,

$\tilde{Y}(t) = (-B(T+t) + B(t))_{t \geq 0}$.

$Y, \tilde{Y}$ are each BM ind. of X.

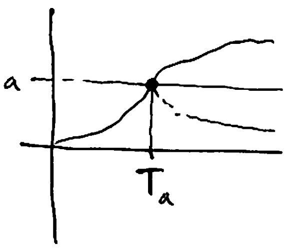
Write $X \circ Y$ for concatenation of processes. Then $\underset{\overset{\parallel}{B}}{X \circ Y}$ has same law as $X \circ \tilde{Y} = \tilde{B}$.

<u>Cor</u>: Let $M(t) := \max_{0\le s\le t} B(t)$. Then, $\forall t$, $\mathrm{Law}(M(t)) = \mathrm{Law}(|B(t)|)$.

<u>Pf</u>: $T_a := \inf\{t : B(t) = a\}$, $\tilde{B}(t)$ = reflection at T_a.

$$\{M(t) > a\} = \{B(t) > a\} \sqcup \{M(t) > a,\ B(t) \le a\}$$

$$\{M(t) > a,\ B(t) \le a\} = \{\tilde{B}(t) \ge a\}$$



$$\Longrightarrow \mathbb{P}[M(t) > a] = \mathbb{P}[B(t) > a] + \mathbb{P}[\tilde{B}(t) \ge a]$$
$$= 2\,\mathbb{P}[B(t) > a]$$
$$= \mathbb{P}[|B(t)| > a].$$

$\forall a \in \mathbb{R}$

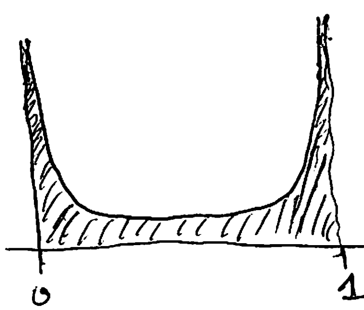
<u>Cor</u>: $\mathbb{P}[T_a \le t] = \mathbb{P}[M(t) \ge |a|] = 2\mathbb{P}[B(t) > |a|] = \frac{2}{\sqrt{2\pi t}} \int_{|a|}^{\infty} \exp\left(-\frac{x^2}{2t}\right) d$

$L := \sup\{0 \le t \le 1 : B(t) = 0\}$ = last zero in $[0,1]$:

$$\mathbb{P}[L \le s] = \mathbb{P}[B(t) \ne 0 \ \forall\ s < t \le 1]$$
$$= \int_{-\infty}^{\infty} \mathbb{P}[T_{-y} > 1 - s]\, d\rho_{s,0}(y) \quad (\text{cond. on } B(s) = y)$$
$$= \int_{-\infty}^{\infty} \mathbb{P}[T_{-y} > 1 - s] \frac{1}{\sqrt{2\pi s}} \exp\left(-\frac{y^2}{2s}\right) dy$$
$$= \frac{2}{\sqrt{2\pi s}} \int_0^{\infty} \underbrace{\mathbb{P}[T_y > 1 - s]}_{= 1 - \mathbb{P}[T_y \le 1-s] = 1 - 2\mathbb{P}[B(1-s) > y]} \exp\left(-\frac{y^2}{2s}\right) dy$$
$$= \dots = \frac{2}{\pi} \arcsin(\sqrt{s}).$$

$\leadsto L$ has density $\frac{1}{\pi\sqrt{s(1-s)}}\, ds$

$\boxed{\text{ARCSINE LAW}}$



Counter-intuitively least likely last zero is at $1/2$!

Recall <u>Donsker's Thm.</u>, vaguely stated previously:

X_i iid w/ $\mathbb{E} X_i = 0$, $\mathbb{E} X_i^2 = 1$, $S(n) := \sum_{i=1}^{n} X_i$,

$$S_n(t) := \begin{cases} \frac{1}{\sqrt{n}} S(tn) & \text{if } tn \in \mathbb{Z} \\ \text{linear interpolation} & \text{otherwise} \end{cases}$$

then $(S_n(t))_{t \in [0,1]} \implies (B(t))_{t \in [0,1]}$.

Formal statement: let $C := \{f : [0,1] \to \mathbb{R} \text{ continuous}\}$ with norm $\|f - g\|_\infty = \sup_{x \in [0,1]} |f(x) - g(x)|$.

<u>Thm</u>: Let $F : C \to \mathbb{R}$ be bounded and continuous w.r.t. topology induced by $\|\cdot\|_\infty$ (i.e., if $\|f_n - f\|_\infty \to 0$, then $|F(f_n) - F(f)| \to 0$).

Then, $\mathbb{E} F(S_n) \longrightarrow \mathbb{E} F(B)$.

Idea of application: take $F(f) := \underbrace{\phi}_{\text{bdd. cts. } \mathbb{R} \to \mathbb{R}.}(\sup \{t \in [0,1] : f(t) = 0\})$

$$\begin{aligned} \implies \mathbb{E} F(S_n) &= \mathbb{E} \phi\left(\sup \{t \in [0,1] : S_n(t) = 0\}\right) \\ &\approx \mathbb{E} \phi\left(\tfrac{1}{n} \cdot \max \{k \in \{1, \dots, n\} : S(k) = 0\}\right) \\ &\to \mathbb{E} F(B) \\ &= \mathbb{E} \phi(Y) \qquad Y \sim \text{arcsine} \end{aligned}$$

<u>General principle</u>: Can get <u>universal</u> info. about random walks by passing to common BM limit.

<u>Thm</u>: $\frac{1}{n} \max \{k \in [n] : S(k) = 0\} \xrightarrow[n \uparrow \infty]{(d)}$ Arcsine.

<u>Rk</u>: F above <u>not</u> continuous! Need to restrict domain to subset of C carefully.