LECTURE 22: last time: BM as a Markov process

Thm: (Weak Markov property) $B(t)$ BM, $s \geq 0 \implies X(t) := B(s+t) - B(s)$ also BM, indep. (in finite distributions) of $(B(t))_{0 \leq t \leq s}$.

Def: Cts. time filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ is $(\mathcal{F}(t))_{t \geq 0}$ σ-algebras s.t. $s \leq t \implies \mathcal{F}(s) \subseteq \mathcal{F}(t) \subseteq \mathcal{F}$.

$X(t)$ adapted if $X(t)$ is $\mathcal{F}(t)$-measurable.

To what filtration is BM adapted? Natural choice?

$$\mathcal{F}^0(t) := \sigma\big(B(s) : 0 \leq s \leq t\big).$$

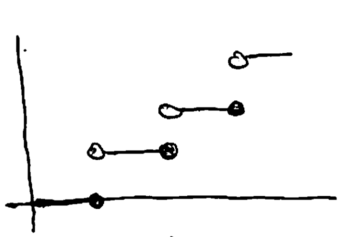
Cor: If $X(t) := B(s+t) - B(s)$, then any $(X(t_1), \ldots, X(t_k))$ is independent of (any $A' \in$) $\mathcal{F}^0(s)$, and so any $A \in \sigma(X(t))$.

Pf: Thm shows for $A = \{B(r_1) \in A_1, \ldots, B(r_m) \in A_m\}$, extend to any A by similar measure theory tools from Markov chain theory.

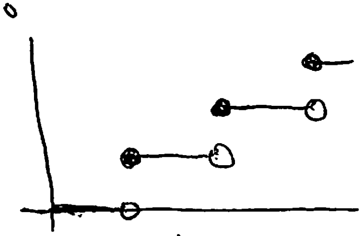
Def: A filtration is right-continuous if $\mathcal{F}(t) = \bigcap_{\varepsilon > 0} \mathcal{F}(t+\varepsilon)$.

Ex: Consider counting process $N(t) \in \mathbb{Z}_{\geq 0}$ like PPP, $\mathcal{F}^0(t) := \sigma(B(s) : 0 \leq s \leq t)$

$\{$there is a jump at time $t\} \in \bigcap_{\varepsilon > 0} \mathcal{F}^0(t+\varepsilon)$. What about $\mathcal{F}^0(t)$?



NO



YES

Right-cts. filtrations "work nicely" w/ right-cts. processes.

<u>Prop</u>: $\mathcal{F}^\circ(t)$ of BM is <u>not</u> right-cts. I.e.,

$$\mathcal{F}^+(t) := \bigcap_{\varepsilon>0} \mathcal{F}^\circ(t+\varepsilon) \supsetneq \mathcal{F}^\circ(t)$$

<u>Pf</u>: $\underbrace{\{\exists\, s_i > t \text{ s.t. } s_i \downarrow t \text{ and } B(s_i) > t\ \forall i\}}_{\{t \text{ not a local maximum}\}} \in \mathcal{F}^+(t) \setminus \mathcal{F}^\circ(t)$

Conventional to work w/ right-cts. filtrations and right-cts. processes.

<u>Thm</u>: (Stronger Markov property) $\forall s \geq 0$, $X(t) := B(s+t) - B(s)$ has all fin. dist. indep. of $\mathcal{F}^+(s)$, and so any event in $\sigma(X(t))$.

<u>Pf Sketch</u>: Pick $s_i \downarrow s$, By continuity

$$\left(B(t_1+s_i) - B(s_i), \ldots, B(t_n+s_i) - B(s_i)\right) \to \text{by Thm, indep. of } \mathcal{F}^\circ(s_i) \supseteq \mathcal{F}^+(s).$$

$$\Big\downarrow \text{(as } i \to \infty)$$

$$\left(B(t_1+s) - B(s), \ldots, B(t_n+s) - B(s)\right)$$

"germ σ-algebra"

<u>Cor</u>: (Blumenthal 0-1 Law) $\forall A \in \mathcal{F}^+(0)$, $\mathbb{P}[A] \in \{0,1\}$.

<u>Pf</u>: $s=0$ in Thm $\Rightarrow$ any $A \in \mathcal{F}^+(0) \subseteq \sigma(B(t))$ indep of itself.

<u>Cor</u>: (Kolmogorov 0-1 Law) $\mathcal{T} := \bigcap_{t\geq 0} \sigma(B(s) : s \geq t) \rightsquigarrow \forall A \in \mathcal{T}, \mathbb{P}[A] \in \{0,1\}$.

"tail σ-algebra"

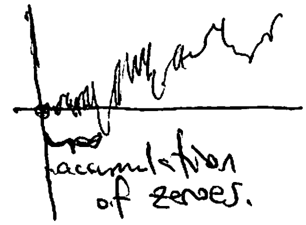
<u>Pf</u>: Time inversion

~~[illegible]~~

<u>Cor</u>: For any $f: \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$,

$$\mathbb{P}\left[\limsup_{t\to 0} \frac{B(t)}{f(t)} \leq 1\right] \in \{0,1\} \quad \text{E.g. last time, } f(t) = t.$$

Cor:
$$T^+ := \inf\{t>0 : B(t)>0\}$$
$$T^- := \inf\{t>0 : B(t)<0\} \quad \rightsquigarrow \quad \text{A.s., } T^+ = T^- = T^0 = 0.$$
$$T^0 := \inf\{t>0 : B(t)=0\}$$



("BM oscillates infinitely often near zero.")

Pf: $\{T^+=0\} = \bigcap_{n\geq 1} \{\exists\, t\in(0,\tfrac{1}{n}) : B(t)>0\} \in \mathcal{F}^+(0)$

$$\mathbb{P}[T^+\leq t] \geq \mathbb{P}[B(t)>0] = \tfrac{1}{2}$$

$$\Longrightarrow \mathbb{P}[T^+=0] = \mathbb{P}\Big[\bigcap_{t>0}\{T^+\leq t\}\Big] = \lim_{t\downarrow 0}\mathbb{P}[T^+\leq t] \geq \tfrac{1}{2}$$

Blumenthal $\Longrightarrow \mathbb{P}[T^+=0]=1$.

Similarly for T^-, and T^0 by intermediate value thm.

<u>Continuous Markov process theory</u>

Def: $p : [0,\infty) \times \mathbb{R} \times \mathcal{B} \longrightarrow [0,1]$ is <u>Markov transition kernel</u> if

$[0,\infty)$: time; $\mathbb{R}$: current position; $\mathcal{B}$: future position; $\mathcal{B}$: Borel σ-alg.

(1) $\forall A\in\mathcal{B}$, $(t,x)\longmapsto p(t,x,A)$ measurable.

(2) $\forall t,x$, $A\longmapsto p(t,x,A)$ is prob. measure, $=: p_{t,x}$

(3) $\forall s,t\in[0,\infty)$, $x\in\mathbb{R}$, $A\in\mathcal{B}$, $p(t+s,x,A) = \int_{\mathbb{R}} p(t,y,A)\, dp_{s,x}(y)$.

Process $(X(t))$ adapted to $(F(t))$ is Markov process w/ kernel p if

$$\mathbb{P}[X(t)\in A \mid F(s)] = p(t-s, X(s), A) \quad \text{a.s.} \quad \forall A,\ t\geq s.$$

Rk: (3) is new to cts. time setting.

Ex: Brownian motion: $p_{t,x} = N(x, t)$, i.e.,

$$p(t, x, A) = \int_A \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y-x)^2}{2t}\right) dy.$$

Ex: PPP(1): only ever have $x \in \mathbb{Z}_{\geq 0}$, $p_{t,x} = x + \text{Pois}(t)$
(Not obviously Markov process!)

Def: Transition operator T_t: functions $\to$ functions,

$$(T_t f)(x) := \mathbb{E}_x f(X(t)) = \int f(y)\, dp_{t,x}(y).$$

Prop: (Semigroup properties)

- $T_0 = \text{identity}$
- $T_s T_t = T_t T_s = T_{s+t}$ (by property (3) of Markov process)

Rk: (T_t) contains all info of kernel, since $(T_t \mathbb{1}_A)(x) = p(t, x, A)$.

Def: (informal) The generator of an MP is the operator

$$(Lf)(x) := \lim_{t \downarrow 0} \frac{(T_t f)(x) - f(x)}{t} = \frac{d}{dt} (T_t f)(x) \Big|_{t=0}.$$

Idea: $\mathbb{E}\left[f(X(t+h)) - f(X(t)) \,\middle|\, \mathcal{F}(t)\right] = (T_h f)(X_t) - f(X_t)$
$= h (Lf)(X_t) + o(h).$

Ex: For BM, $(T_t f)(x) = \mathbb{E}_{g \sim N(0,t)} f(x+g) \approx \mathbb{E}\left[f(x) + g f'(x) + \frac{g^2}{2} f''(x) + O(g^3)\right]$
$= f(x) + \frac{f''(x)}{2} t + O(t^{3/2})$

$\Longrightarrow (Lf)(x) = \frac{1}{2} f''(x)$

$f(0,x) := f(x)$, $f(t,x) := \mathbb{E}_x f(B(t)) \rightsquigarrow \boxed{\partial_t f = \frac{1}{2} \partial_x^2 f}$

HEAT EQUATION.