① LECTURE 21 : distributional symmetries of Brownian motion:

Prop: (Scaling) If $B(t)$ is a BM, $c>0$, then $X(t) := \frac{1}{c} B(c^2 t)$ is also a BM.

Pf: Check the properties:

① $X(0) = \frac{1}{c} B(0) = 0$.

② X continuous.

③ Finite distributions: $\mathrm{Law}(X(t)-X(s)) = \mathrm{Law}\Big(\frac{1}{c}\underbrace{[B(c^2t) - B(c^2 s)]}_{N(0,\, c^2t - c^2 s)}\Big)$

$= N(0, t-s)$, similarly for products.

Alternatively: observe X also Gaussian process, so enough to check that means + covariances match:

$$\mathbb{E} X(t) = \frac{1}{c} \mathbb{E} B(c^2 t) = 0 = \mathbb{E} B(t)$$

$$\mathbb{E} X(s) X(t) = \frac{1}{c} \mathbb{E} B(c^2 s) B(c^2 t) = \frac{1}{c} \min\{c^2 s, c^2 t\}$$

$$= \min\{s, t\} = \mathbb{E} B(s) B(t).$$

Prop: (Time inversion) $B(t)$ BM $\Longrightarrow X(t) := \begin{cases} 0 & \text{if } t=0 \\ tB(\frac{1}{t}) & \text{if } t>0 \end{cases}$ BM.

Pf: ① $X(0) = 0$ by def.

② GP method:

$$\mathbb{E} X(t) = t\, \mathbb{E} B\left(\tfrac{1}{t}\right) = 0$$

$$\mathbb{E} X(s) X(t) = st\, \mathbb{E} B\left(\tfrac{1}{s}\right) B\left(\tfrac{1}{t}\right) = st \min\left\{\tfrac{1}{s}, \tfrac{1}{t}\right\}$$

$$= \frac{st}{\max\{s,t\}} = \min\{s,t\}.$$

③ For all $t>0$, $X(t)$ cts. a.s. since $B(t)$ cts.

Remains to show $X(t)$ cts. at 0 a.s.

$$\left\{\lim_{t\downarrow 0} X(t) = 0\right\} = \left\{\forall \varepsilon > 0\ \exists \delta > 0 \text{ s.t. } |X(t)| \le \varepsilon\ \forall |t| \le \delta\right\}$$

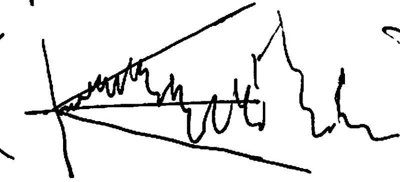
$$= \left\{\forall m > 0\ \exists n > 0 \text{ s.t. } |X(t)| \le \tfrac{1}{m}\ \forall t \in \mathbb{Q} \cap (0, \tfrac{1}{n})\right\}$$

$$= \bigcap_{m \ge 1} \bigcup_{n \ge 1} \bigcap_{t \in \mathbb{Q}\cap(0,\frac{1}{n})} \left\{|X(t)| \le \tfrac{1}{m}\right\}$$

Enough to show $\lim_{n\to\infty} \mathbb{P}\Big[\bigcap_{t \in \underbrace{\mathbb{Q}\cap(0,\frac{1}{n})}_{=\{t_{n,1},\, t_{n,2},\, \dots\}}} \{|X(t)| \le \tfrac{1}{m}\}\Big] = 1\ \forall m \ge 1.$

$$\boxed{\text{using finite distributions}}\quad = \lim_{n\to\infty} \lim_{k\to\infty} \mathbb{P}\left[|X(t_{n,1})|, \dots, |X(t_{n,k})| \le \tfrac{1}{m}\right]$$

$$= \lim_{n\to\infty} \lim_{k\to\infty} \mathbb{P}\left[|B(t_{n,1})|, \dots, |B(t_{n,k})| \le \tfrac{1}{m}\right] = 1. \blacksquare$$

<u>Cor</u>: Almost surely, $\frac{B(t)}{t} \to 0$. ()

<u>Thm</u>: $\forall \varepsilon > 0$, almost surely, $\frac{B(t)}{t^{1/2+\varepsilon}} \to 0$. (Harder.)

<u>Application</u>: Exit time scaling: recall if $S(n)$ = SRW on $\mathbb{Z}$, $-\overset{\in\mathbb{Z}}{a} < 0 < \overset{\in\mathbb{Z}}{b}$, $T(-a,b) := \min\{n \ge 0 : S(n) \notin [-a,b]\}$ then we showed w/ clever martingales that $\mathbb{E}T(-a,b) = ab$.

<u>Cor</u>: $-\overset{\in\mathbb{R}}{a} < 0 < \overset{\in\mathbb{R}}{b}$, $T(-a,b) = \inf\{t \ge 0 : B(t) \notin [-a,b]\}$.

Then $\mathbb{E}T(-a,b) = a^2\, \mathbb{E}T(-1, \tfrac{b}{a})$, $\mathbb{E}T(-b,b) = b^2 \underbrace{\mathbb{E}T(-1,1)}_{\text{const.}}$.

<u>Pf</u>: $X(t) := aB(t/a^2)$ is BM.

$$\mathbb{E}T(-a,b) = \mathbb{E}\inf\{t \ge 0 : X(t) \notin [-a,b]\}$$

$$= a^2\, \mathbb{E}\inf\{t \ge 0 : aB(t) \notin [-a,b]\} = a^2\, \mathbb{E}T(-1, \tfrac{b}{a}).$$

(3) **Brownian Motion as a Markov Process:**

Idea: $(B(t))_{t\geq s}$ looks like BM started from $B(s)$, and cond. on $B(s)$ is independent of $(B(t))_{0\leq t\leq s}$.

Rk: same for bounded times $(X(t))_{t\in[a,b]}$.

Def: Cts. time processes $(X(t))_{t\geq 0}$, $(Y(t))_{t\geq 0}$ are independent if finite distributions are independent: $\forall s_1, \ldots, s_m, t_1, \ldots, t_n$,
$(X(s_1), \ldots, X(s_m))$ indep. of $(Y(t_1), \ldots, Y(t_n))$.

Thm: (Weak Markov property) Let $(B(t))_{t\geq 0}$ be a BM and $s\geq 0$. Then, $X(t) := B(t+s) - B(s)$ is BM indep. of $(B(t))_{0\leq t\leq s}$.

Pf: $X(0) = 0$, X cts. Exc: check finite distributions.

Def: Cts. time filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ is $(\mathcal{F}(t))_{t\geq 0}$ σ-algebras s.t. $\mathcal{F}(s) \subseteq \mathcal{F}(t) \subseteq \mathcal{F}$ $\forall s \leq t$. $(X(t))$ adapted if $X(t)$ $\mathcal{F}(t)$-m'able.

Natural filtration for BM? $\mathcal{F}^0(t) := \sigma(B(s) : 0 \leq s \leq t)$.

Cor: $X(t)$ from Thm is indep. of $\mathcal{F}^0(s)$. Pf: Thm + measure theory approx. argument.

Def: A filtration is right-cts. if $\mathcal{F}(t) = \bigcap_{\varepsilon>0} \mathcal{F}(t+\varepsilon)$. For BM,
$\mathcal{F}^+(t) := \bigcap_{\varepsilon>0} \mathcal{F}^0(t+\varepsilon)$. ("infinitesimal glance into future")

Prop: $\mathcal{F}^+(t)$ is right-cts., but $\mathcal{F}^0(t)$ is not; $\mathcal{F}^+(t) \supsetneq \mathcal{F}^0(t)$.

Pf: If $\mathcal{F}^0(t)$ were right-cts., would have $\mathcal{F}^0(t) = \mathcal{F}^+(t)$. But, e.g.,
$\{t \text{ not local max of } B(t)\} = \{\exists s_i \to t \text{ s.t. } B(s_i) > B(t)\} \in \mathcal{F}^+(t) \setminus \mathcal{F}^0(t)$.