(1)

LECTURE 20:

Brownian motion : $(B(t))_{t\geq 0}$, like PPP is continuous-time process, only now $B(t) \in \mathbb{R}$: continuous time and value.

Reminders of setup:

- Underlying $(\Omega, \mathcal{F}, \mathbb{P})$, require $B(t): \Omega \longrightarrow \mathbb{R}$ r.v. $\forall t \geq 0$.
- $\mathcal{S}^{(0)} := \{f : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}\}$, then $B: \Omega \longrightarrow \mathcal{S}^{(0)}$ is measurable if give σ-algebras $(\Omega, \mathcal{F})$, $(\mathcal{S}^{(0)}, \mathcal{B}^{\mathbb{R}_{\geq 0}})$ → product alg. of Borel $=: \mathcal{G}$
- Events like $\{\omega : B \text{ is continuous}\}$ are not measurable, $\notin \mathcal{G}$
- w.r.t product alg., Law(B) = {finite distributions of $(B(t_1), \ldots, B(t_k))$}
- "Hack" to enhance measurable events: force $B: \Omega \longrightarrow \mathcal{S} \subset \mathcal{S}^{(0)}$, e.g. $\mathcal{S}^{(c)} := \{f : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}\}$ continuous.

Thm: (Kolmogorov continuity) Let $B: \Omega \longrightarrow \mathcal{S}^{(0)}$ be $\mathcal{G}$-measurable and have that $\forall T > 0$, $\exists\, C, \alpha, \beta > 0$ s.t.

$$\mathbb{E}\,|B(t) - B(s)|^{\alpha} \leq C\,|t-s|^{1+\beta} \quad \forall s,t \in [0,T].$$

Then $\exists\, \tilde{B}: \Omega \longrightarrow \mathcal{S}^{(c)}$ s.t. $\forall t \geq 0$, $\mathbb{P}[\tilde{B}(t) = B(t)] = 1$

($\tilde{B}$ version/modification of B.)

In particular, Law$(\tilde{B})$ = Law(B), i.e., same finite distributions.

Def: $B: \Omega \longrightarrow \boxed{\mathcal{S}^{(c)}}$ is Brownian motion if it satisfies $B(0) = 0$ a

$$\text{Law}\big((B(t_1), B(t_2) - B(t_1), \ldots, B(t_k) - B(t_{k-1}))\big)$$
$$= \mathcal{N}(0,t_1) \otimes \mathcal{N}(0, t_2 - t_1) \otimes \cdots \otimes \mathcal{N}(0, t_k - t_{k-1}).$$

Thm: (Existence) $\exists (\Omega, \mathcal{F}, \mathbb{P})$ and $B: \Omega \to \mathcal{S}^{(c)}$ that is Brownian motion

Pf: $B : \Omega \to \boxed{\mathcal{S}^{(s)}}$ with correct finite distributions exists by Kolmogorov extension thm. Use K. continuity to build $\tilde{B} : \Omega \to \mathcal{S}^{(c)}$:

$$\mathbb{E}|B(t) - B(s)|^{\alpha} = \underset{\Delta \sim \mathcal{N}(0,t-s)}{\mathbb{E}|\Delta|^{\alpha}} = |t-s|^{\alpha/2} \underset{\Delta \sim \mathcal{N}(0,1)}{\mathbb{E}|\Delta|} = \sqrt{\tfrac{2}{\pi}}\,|t-s|^{\alpha/2}$$

$\rightsquigarrow$ choose any $\alpha > 2$.

Rk: "Arbitrary" events concerning $(B(q))_{q \in \mathbb{Q}_{\geq 0}}$ are $\mathcal{G}$-measurable, and can ask other questions about B using guarantee of continuity.

Rk: One motivation: X_i iid w/ $\mathbb{E}X_i = 0$, $\mathrm{Var}(X_i) = 1$, $S(n) := \sum_{i=1}^{n} X_i$

CLT: $\frac{1}{\sqrt{n}} S(n) \xrightarrow{(d)} \mathcal{N}(0,1)$.

Consider: $B^{(n)}(t) := \begin{Bmatrix} \frac{1}{\sqrt{n}} S(tn) & \text{if } tn \in \mathbb{Z} \\ \text{linear interp} & \text{otherwise} \end{Bmatrix}$.

Cf: Donsker Thm. for stronger mode of conv.

$$(B^{(n)}(t))_{t \in [0,1]} \xrightarrow{(f.d.)} (B(t))_{t \in [0,1]}$$

BM = scaling limit of random path of RW.

Basic Properties:

Rk: Non-trivial analysis fact that such functions exist! [Weierstrass, 1872].

Thm: $\mathbb{P}[B \text{ nowhere differentiable}] = 1$.

Pf: B differentiable at $t \Rightarrow \lim_{h \downarrow 0} \frac{B(t+h) - B(t)}{h}$ exists

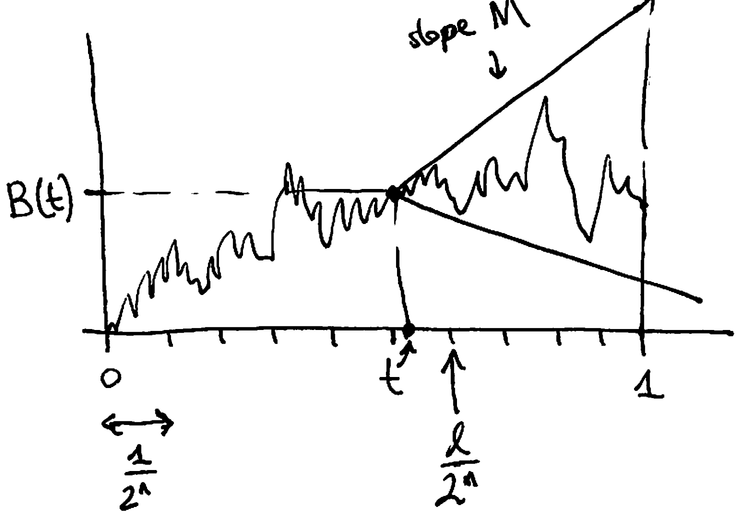
$$\Rightarrow \sup_{h \in [0,1]} \frac{|B(t+h) - B(t)|}{h} < \infty.$$

$$E_{M,k} = \left\{ \exists\, t \in [k, k+1] : \sup_{h \in [0,1]} \frac{|B(t+h) - B(t)|}{h} \leq M \right\}.$$

$\{B$ somewhere differentiable$\}$ in $\bigcup_{k \geq 0} \bigcup_{M > 0} E_{M,k}$

Will show $\mathbb{P}[E_{M,k}] = 0 \;\; \forall M > 0, k \geq 0$.

(3) Will show for $k=0$, same argument for general k.



B diff'able at t, $t \in \left[\frac{l-1}{2^n}, \frac{l}{2^n}\right]$

$$\Downarrow$$

$$\begin{aligned}
&\left|B\left(\tfrac{l+j}{2^n}\right) - B\left(\tfrac{l+j-1}{2^n}\right)\right| \\
&\le \left|B\left(\tfrac{l+j}{2^n}\right) - B(t)\right| + \left|B\left(\tfrac{l+j-1}{2^n}\right) - B(t)\right| \\
&\le M \cdot \left[\frac{l+j}{2^n} - \frac{l-1}{2^n}\right] + M \cdot \left[\frac{l+j-1}{2^n} - \frac{l-1}{2^n}\right] \\
&= M \cdot \frac{2j+1}{2^n} \quad \forall j = 1, 2, \dots, 2^n - 1.
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}[E_{M,0}] &\le \mathbb{P}\left[\exists\, l \in \{1, \dots, 2^n\} : \left|B\left(\tfrac{l+j}{2^n}\right) - B\left(\tfrac{l+j-1}{2^n}\right)\right| \le M \cdot \tfrac{2j+1}{2^n} \quad \forall j = 1, 2, \dots, 2^{n-1}\right] \\
&\overset{\text{union bound}}{\le} \sum_{l=1}^{2^n} \mathbb{P}\Big[\big|\underbrace{B\left(\tfrac{l+j}{2^n}\right) - B\left(\tfrac{l+j-1}{2^n}\right)}_{\overset{(d)}{=} \mathcal{N}(0, \frac{1}{2^n}),\ \text{indep for different } j}\big| \le \frac{7M}{2^n} \quad \forall j = 1, 2, 3\Big] \\
&= 2^n \cdot \Big(\underset{\Delta \sim \mathcal{N}(0,1)}{\mathbb{P}}\left[|\Delta| \le \frac{7M}{2^{n/2}}\right]\Big)^3 \le 2^n \cdot \left(\frac{7M}{2^{n/2}}\right)^3 \lesssim 2^{-n/2} \to 0.
\end{aligned}$$

∎

Intuition: $|B(t+h) - B(t)| \asymp \sqrt{h}$ since $\mathcal{N}(0,h)$, while $= O(h)$ if differentiable. Similarly:

Thm: On any $[0,T]$, B has unbounded variation: almost surely,

$$\sup_{0 = t_0 < t_1 < \dots < t_k = T} \sum |B(t_i) - B(t_{i-1})| = \infty.$$

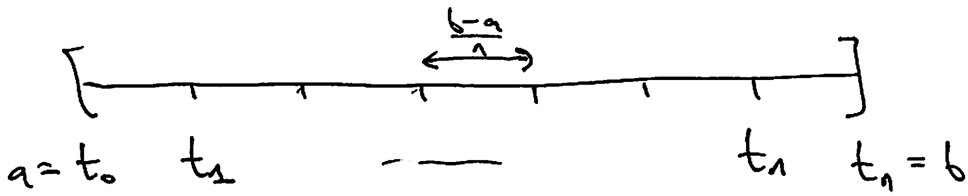
<u>Thm</u>: Almost surely, $\forall\, 0 < a < b$, $B(t)$ <u>not</u> monotone on $[a,b]$.

<u>Pf</u>: (1) Reduce to countable union:

$$\{B(t) \text{ monotone on some } [a,b]\}$$
$$= \bigcup_{\substack{0<a<b \\ a,b\in\mathbb{R}}} \{B(t) \text{ monotone on } [a,b]\}$$
$$= \bigcup_{\substack{0<a<b \\ a,b\in\mathbb{Q}}} \underbrace{\{B(t) \text{ monotone on } [a,b]\}}_{=:\, E_{a,b}}$$

Enough to show $\mathbb{P}[E_{a,b}] = 0 \ \forall\, a<b \in \mathbb{Q}$.

(2) Reduce to finite distributions: introduce some $n \geq 1$.



$$\mathbb{P}[E_{a,b}] \leq \mathbb{P}[B(t_0) \leq \cdots \leq B(t_n) \text{ OR } B(t_0) \geq \cdots \geq B(t_n)]$$
$$\leq \mathbb{P}[B(t_0) \leq \cdots \leq B(t_n)] + \mathbb{P}[B(t_0) \geq \cdots \geq B(t_n)]$$
$$= \mathbb{P}[B(t_i) - B(t_{i-1}) \geq 0 \ \forall i \in [n]]$$
$$+ \mathbb{P}[B(t_i) - B(t_{i-1}) \leq 0 \ \forall i \in [n]]$$
$$= 2 \cdot \left(\tfrac{1}{2}\right)^n \quad \leftarrow \text{by independence + Gaussianity of increments.}$$

$n \to \infty \implies \mathbb{P}[E_{a,b}] = 0$.