LECTURE 18:

Thm: (Uniqueness of stationary measure) If S irreducible, all states recurrent $\Rightarrow \exists!$ non-zero stationary μ up to rescaling. (i.e. all $= c\mu$)

Pf: let $\mu \neq 0$ stationary, $x \in S$ w/ $\mu(x) \neq 0$. WLOG $\mu(x) = 1$, will show then $\mu = \mu_x$ (recall $\mu_x(x) = 1$).

Will see below:
($\mu_x(y) > 0 \ \forall y$; $\mu(y) > 0$ also.)

Consider some $y \neq x$:

$$\mu(y) = \sum_{z_0} \mu(z_0) p(z_0, y) = \underbrace{p(x,y)}_{\mathbb{P}_x[Y_1 = y,\, T_x > 1]} + \underbrace{\sum_{z_0 \neq x} \mu(z_0) p(z_0, y)}_{\star}$$

$$\star = \sum_{z_0 \neq x,\, z_1} \mu(z_1) p(z_1, z_0) p(z_0, y) = \underbrace{\sum_{z_0 \neq x} p(x, z_0) p(z_0, y)}_{\mathbb{P}_x[Y_2 = y,\, T_x > 2]} + \underbrace{\sum_{z_0, z_1 \neq x} \mu(z_1) p(z_1, z_0)\, p(z_0, y)}_{\star}$$

... repeat $\Rightarrow \forall N,\ \mu(y) \geq \sum_{n=1}^{N} \mathbb{P}_x[Y_n = y,\, T_x > n]$

$$\Rightarrow \mu(y) \geq \sum_{n=1}^{\infty} \mathbb{P}_x[Y_n = y,\, T_x > n] = \mu_x(y) \Rightarrow \boxed{\mu(y) \geq \mu_x(y)}$$

$\mu' := \mu - \mu_x$ stationary, $\mu'(x) = 0$.

But, S irreducible $\Rightarrow \forall y,\ \rho_{yx} > 0 \Rightarrow p^{(k)}(y,x) > 0$ for some k.

$$0 = \mu'(x) = \sum_{z} \mu'(z) p^{(k)}(z, x) \geq \mu'(y) p^{(k)}(y, x) \Rightarrow \mu'(y) = 0 \quad \forall y$$

$\Rightarrow \mu' = 0 \Rightarrow \mu = \mu_x$. ■

Cor: S irreducible + recurrent, $\mu \overset{\neq 0}{} $ stationary $\Rightarrow \mu(x) > 0 \ \forall x$.

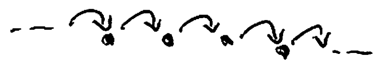
②

Lem: (Partial converse) If μ stationary prob measure, $\mu(x) > 0 \implies x$ recurrent.

Pf: $\forall n$, $\mu(x) = \sum_y \mu(y) p^{(n)}(y,x)$.

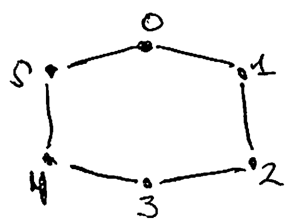
OTOH, $\sum_{n=1}^{\infty} p^{(n)}(x,y) = \sum_{n=1}^{\infty} \mathbb{P}_x[Y_n = y] = \mathbb{E}_x \underbrace{N(y)}_{\text{(\# visits to } y)} = \dfrac{\rho_{xy}}{1-\rho_{yy}}$

$$\infty = \sum_{n=1}^{\infty} \mu(x) = \sum_y \mu(y) \frac{\rho_{xy}}{1-\rho_{yy}} \leq \frac{1}{1-\rho_{yy}} \implies \rho_{yy} = 1.$$

Rk:  example shows μ must be finite.

Convergence: recall, $\mu_{x,n}(y) := \mathbb{P}_x[Y_n = y]$. If μ stationary, $\mu_{x,n}(y) \to \mu(y)$?

One obstruction: periodicity:

Ex: 

SRW on this graph $\rightsquigarrow$ $\begin{cases} \mathbb{P}_0[Y_{2n} = 1] = 0, \\ \mathbb{P}_0[Y_{2n+1} = 1] \to \frac{1}{3}. \end{cases}$

$\mu(x) \equiv \frac{1}{6}$ stationary,

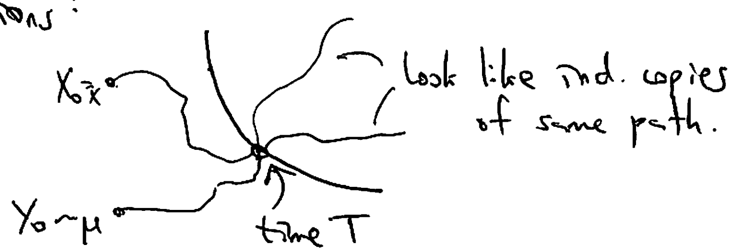
Def: MC aperiodic if $\forall x \in S^{\text{recurrent}}$ $\exists n_0$ s.t. $p^{(n)}(x,x) > 0$ $\forall n \geq n_0$.

Exc: Aperiodic $\iff \forall x \in S^{\text{recurrent}}$, there is no $k \geq 2$ divisor of all $\{n : p^{(n)}(x,x) > 0\}$.

Thm: Suppose MC is recurrent + irreducible + aperiodic, and μ is stationary prob. measure. Then, $\forall x \in S$, $\mu_{x,n} \xrightarrow[n\to\infty]{\text{(weak)}} \mu$.

Cor: same weak convergence for $\nu_n(y) = \mathbb{P}_{\mu_0}[Y_n = y]$ $\forall \mu_0$.

③ <u>Pf idea</u>: <u>Coupling</u>: compare (X_n) started from $X_0 = x$ versus (Y_n) from $Y_0 \sim \mu_0$. Observations:



(1) $\mathrm{Law}(X_n) = \mu_{x,n}$

(2) $\mathrm{Law}(Y_n) = \mu$

(3) $T := \min\{n \geq 1 : X_n = Y_n\} \rightsquigarrow$ "after T look identical":

$$\mathrm{Law}(X_{T+n}) = \mathrm{Law}(Y_{T+n}) = \mu.$$

<u>Pf</u>: Define kernel on S^2: $q((x_1,y_1),(x_2,y_2)) = p(x_1,x_2)\,p(y_1,y_2)$.

- q <u>irreducible</u>: p irred. $\Longrightarrow p^{(k)}(x_1,x_2),\ p^{(\ell)}(y_1,y_2) > 0$.

  aperiodic $\Longrightarrow \exists\, m$ s.t. $p^{(\ell+m)}(x_2,x_2),\ p^{(k+m)}(y_2,y_2) > 0$.

$$\Longrightarrow q^{(k+\ell+m)}((x_1,y_1),(x_2,y_2)) \geq p^{(k)}(x_1,x_2)\,p^{(\ell+m)}(x_2,x_2)\,p^{(\ell)}(y_1,y_2)\,p^{(k+m)}(y_2,y_2) > 0.$$

- $\nu((x,y)) := \nu(x)\,\nu(y)$ <u>stationary</u> for q, and prob. measure.
- Lem $\Longrightarrow$ all states recurrent for q. (S recurrent $\Longrightarrow \mu > 0$; $\nu = \mu^{\otimes 2}$)

Let (X_n, Y_n) be MC w/ transition kernel q, init. from $\delta_x \otimes \mu$.

<u>Claim</u>: On $\{T \leq n\}$, X_n and Y_n have same law. I.e.:

$$\mathbb{P}[X_n = z, T \leq n] = \sum_{m=0}^{n} \sum_{y} \mathbb{P}[T = m, X_m = y, X_n = z]$$

$$= \sum_{m=0}^{n} \sum_{y} \mathbb{P}[T = m, \underset{(Y_m)}{X_m} = y] \cdot \mathbb{P}[X_n = z \mid X_m = y, \cancel{T = m}] \quad \text{Markov prop.}$$

$$= \sum_{m=0}^{n} \sum_{y} \mathbb{P}[T = m, Y_m = y] \cdot \mathbb{P}[Y_n = z \mid Y_m = y] = \dots = \mathbb{P}[Y_n = z, T \leq n].$$

Finally, compare distributions of interest:

$$\begin{aligned}\mu_{x,n}(y) &= \mathbb{P}[X_n = y] \\ &= \mathbb{P}[X_n = y, T \le n] + \mathbb{P}[X_n = y, T > n] \\ &= \mathbb{P}[Y_n = y, T \le n] + \mathbb{P}[X_n = y, T > n] \\ &= \mu(y) - \mathbb{P}[Y_n = y, T > n] + \mathbb{P}[X_n = y, T > n].\end{aligned}$$

$$\Longrightarrow \left|\mu_{x,n}(y) - \mu(y)\right| \le \mathbb{P}[X_n = y, T > n] + \mathbb{P}[Y_n = y, T > n]$$

$$\Longrightarrow \sum_y \left|\mu_{x,n}(y) - \mu(y)\right| \le 2\,\mathbb{P}[T > n]$$

Exc: Condition on starting $y \sim \mu$.

S^2 recurrent + irreducible $\Longrightarrow \mathbb{P}_{\delta_x \otimes \mu}[\text{ever reach } (z,z)] = 1 \Longrightarrow T < \infty$ a.s.

$$\Longrightarrow \sum_y \left|\mu_{x,n}(y) - \mu(y)\right| \xrightarrow[n \uparrow \infty]{} 0$$

Rk: $\mu(y) > 0 \ \forall y$ by earlier Cor.

$$\sum_y \left|\frac{\mu_{x,n}(y)}{\mu(y)} - 1\right| \mu(y) \geq d_1(\mu, \mu_{x,n}) \quad \text{(HW 2, Prob 4)}$$

= "total variation" distance.

$$\Longrightarrow \mu_{x,n} \xrightarrow{\text{(weak)}} \mu. \quad \blacksquare$$

Rk: Important applications in Markov Chain Monte Carlo $\longrightarrow$ given μ, design p making μ stationary.