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LECTURE 17: End of last time:

Thm: (MC structure thm) $x \longleftrightarrow y$ if $p_{xy}, p_{yx} > 0$ or $x = y$.

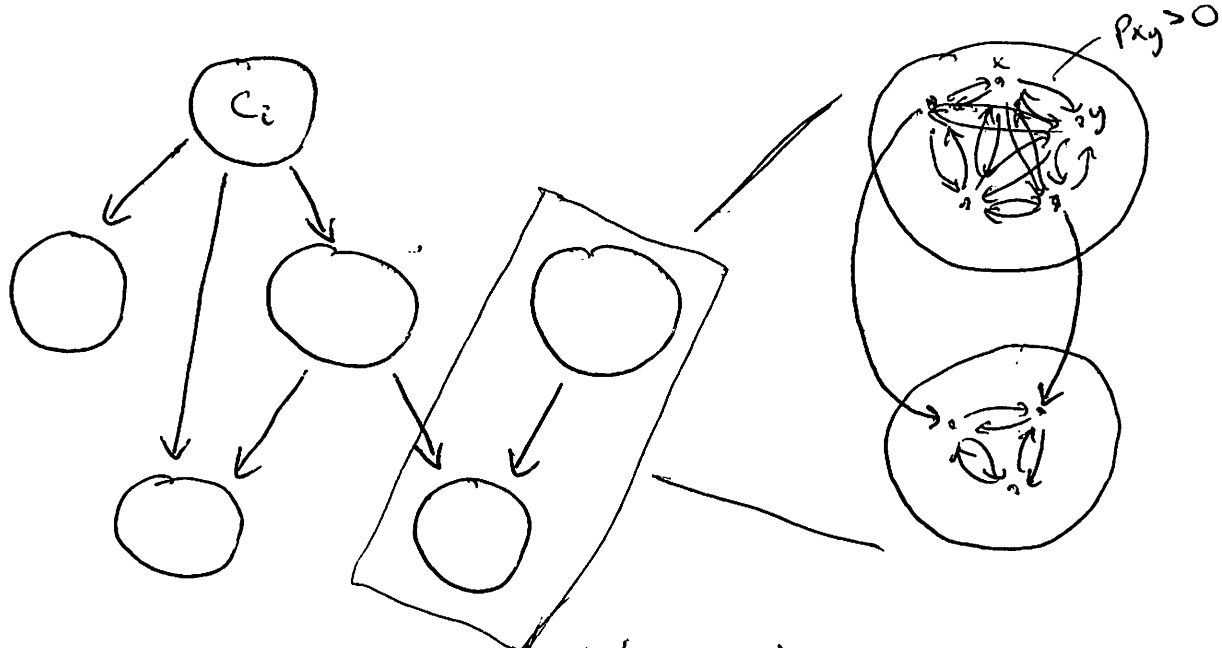
$\{C_i\} :=$ equiv. classes of $\longleftrightarrow$, $S = C_1 \sqcup C_2 \sqcup \cdots$.

Draw directed graph G on $\{C_i\}$: $C_i \to C_j$ if $\exists x \in C_i, y \in C_j : p_{xy} > 0$

(1) G is a DAG.

(2) Every R_x is a leaf. ($\Longrightarrow$ all recurrent states in leaves.)

(3) C_i is closed iff C_i is leaf.



High-level summary of accessibility structure.

Next Q: Given μ_0 and p, write $\mu_n = \text{Law}(Y_n)$ (when $Y_0 \sim \mu_0$).

$$\mu_n(x) := \mathbb{P}_{\mu_0}[Y_n = x] \overset{\text{def. of MC}}{=} \sum_y \mu_0(y) p^{(n)}(y,x)$$

$$= \sum_{y,z} \mu_0(y) p^{(n-1)}(y,z) p(z,x) = \sum_z \mu_{n-1}(z) p(z,x)$$

Matrix notation: If $|S| < \infty$, $P \in [0,1]^{S \times S}$, $\mu_n^T = \mu_0^T P^n = \mu_{n-1}^T P$.

(2) <u>Q</u>: Do $\mu_n \longrightarrow \mu_\infty$? Such object should be --

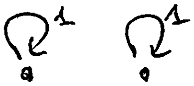
<u>Def</u>: μ is <u>stationary</u> prob. measure if $\sum_y \mu(y)\, p(y,x) = \mu(x)\ \forall x \in S$.

<u>Prop</u>: If μ stationary prob. measure, $\mu_0 = \mu \Longrightarrow \mu_n = \mu\ \forall n \geq 0$.

<u>Rk</u>: If $|S| < \infty$, equiv. to μ^T is left eigvec of P w/ eigval $= 1$.

<u>Ex</u>: (Y_n) = SRW on $\mathbb{Z} \longrightarrow \mu(\{x\}) = 1\ \forall x$ is stationary.

But, $\nexists$ stationary prob. measure: need $\mu(x) = \dfrac{\mu(x+1) + \mu(x-1)}{2}$.

<u>Ex</u>:  has many stationary measures — all are!

Generally, any convex comb. of stationary measures on <u>closed</u> subsets.

<u>Ex</u>: $0 \xrightarrow{1} 1 \xrightarrow{1} 2 \xrightarrow{1} 3 \xrightarrow{1} 4 \cdots$ $\qquad \mu(0) = \sum_y \mu(y) \underbrace{p(y,0)}_{=0} = 0$

But, two-sided version does have $\mu(x) \equiv 1$ stationary. $\qquad \mu(x) = \mu(x-1) \Longrightarrow \mu \equiv 0$.

<u>Existence</u>:

<u>Rk</u>: = amount of time @ y in one "tour", but since x recurrent, really $\propto$ total time @ y.

<u>Thm</u>: Suppose x recurrent. $T_x := \min\{n \geq 1 : Y_n = x\}$.

$$\mu_x(y) := \mathbb{E}_x \sum_{n=0}^{T_x - 1} \mathbb{1}\{Y_n = y\} = \sum_{n \geq 0} \underbrace{\mathbb{P}_x[Y_n = y,\ T_x > n]}_{=:\, q_n(x,y)}$$

<u>Rk</u>: • $\mu_x(y) < \infty$.

• When finite? $\mu_x(S) = \sum_y \mu_x(y) = \mathbb{E}_x T_x$. ⤷ "dominated" by geometric — see HW.

<u>Exc</u>: $x = 0$ in SRW $\longrightarrow x$ recurrent, but $\mu_x(S) = \infty$.

• $\mu_x(y) > 0 \Longrightarrow \rho_{xy} > 0 \Longrightarrow y \in R_x$, also recurrent.

<u>Pf</u>: $$\sum_y \mu_x(y)\, p(y,z) = \sum_{n \geq 0} \sum_y q_n(x,y)\, p(y,z) \overset{(?)}{=} \mu_x(z) = \sum_{n \geq 0} q_n(x,z).$$

(1) $\underline{z \neq x}$: $\sum_y q_n(x,y)\, p(y,z) = \sum_y \mathbb{P}_x[Y_n = y, T_x > n, Y_{n+1} = z]$

$= \mathbb{P}_x[Y_{n+1} = z, T_x > n+1] = q_{n+1}(x,z)$,

and note $q_0(x,z) = 0$ as $z \neq x$.

(2) $\underline{z = x}$: Note $\mu_x(x) = \mathbb{E}_x \sum_{n=0}^{T_x - 1} \mathbb{1}\{Y_n = x\} = 1$.

$\sum_y q_n(x,y)\, p(y,x) = \sum_y \mathbb{P}_x[Y_n = y, T_x > n, Y_{n+1} = x]$

$= \mathbb{P}_x[T_x = n+1]$,

and note $\mathbb{P}[T_x = 0] = 0$.

<u>Uniqueness:</u>

<u>Thm</u>: If S is irreducible and all states recurrent ($S = R_x$), then stationary measure $\neq 0$ unique up to rescaling.

<u>Ex</u>: SRW on $\mathbb{Z}$: $\mu(x) \equiv 1$ unique stationary measure

$\Longrightarrow \mu_0(y) = \mathbb{E}_0[\text{visits to } y \text{ before return}] = \underset{1.}{\text{const.}}\ \forall y$!

<u>Pf</u>: Let $\mu \neq 0$ stationary, $x \in S$ recurrent with $\mu(x) > 0$.
WLOG by rescaling, $\mu(x) = 1$. Will show $\mu = \mu_x$.
Enough to consider $y \neq x$:

$$\mu(y) = \sum_{z_0} \mu(z_0)\, p(z_0, y) = \underbrace{p(x,y)}_{\mathbb{P}_x[Y_1 = y,\, T_x > 1]} + \overbrace{\sum_{z_0 \neq x} \mu(z_0)\, p(z_0, y)}^{\circledast}$$

$$\circledast = \sum_{z_0 \neq x,\, z_1} \mu(z_1)\, p(z_1, z_0)\, p(z_0, y) = \underbrace{\sum_{z_0 \neq x} p(x, z_0)\, p(z_0, y)}_{\mathbb{P}_x[Y_2 = y,\, T_x > 2]} + \sum_{z_0, z_1 \neq x} \mu(z_1)\, p(z_1, z_0)\, p(z_0, y)$$

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Repeating $\Rightarrow \forall n,\ \mu(y) \geq \sum_{n=1}^{N} \mathbb{P}[Y_n = y, T_x > n] \Rightarrow \mu(y) \geq \mu_x(y)$

$\mu' := \mu - \mu_x$ is:

- stationary
- $\mu'(y) \geq 0$ (measure)
- $\mu'(x) = \mu(x) - \mu_x(x) = 1 - 1 = 0.$

But, $\forall y,\ f_{yx} > 0 \Rightarrow p^{(k)}(y,x) > 0$ for some k.

$$0 = \mu'(x) = \sum_z \mu'(z) p^{(k)}(z,x) \geq \mu'(y) p^{(k)}(y,x) \Rightarrow \mu'(y) = 0$$

$$\Rightarrow \mu' = 0 \Rightarrow \mu = \mu_x. \blacksquare$$

<u>Linear Algebra Perspective</u>: $|S| < \infty$, $P \in [0,1]^{S \times S}$.

P transition kernel $\longleftrightarrow$ rows prob. measures $\Longleftrightarrow$ $P\mathbb{1} = \mathbb{1}$ ("stochastic")

S irreducible $\longleftrightarrow$ $\forall x,y\ \exists k$ s.t. $(P^k)_{xy} > 0$.

(e.g., true if $P_{xy} > 0\ \forall x,y$.)

<u>Cor</u>: If $P_{xy} > 0$, P stochastic, then $\exists!\ \mu \geq 0$, $\sum \mu_i = 1$ s.t. $\mu^T P = \mu$ (left eigenvector w/ eigval 1.)

Stronger statement: <u>Perron-Frobenius Thm</u>.