LECTURE 16: Important notions from last time:

(Y_n) MC on countable $S \rightsquigarrow \rho_{xy} := \mathbb{P}_x[Y_n = y \text{ for some } n \geq 1]$

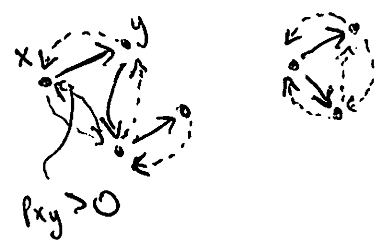
$$N(y) := \sum_{n \geq 1} \mathbb{1}\{Y_n = y\}$$

x <u>recurrent</u> $\longleftrightarrow \rho_{xx} = 1 \longleftrightarrow \mathbb{E}_x N(x) = \infty$,

x <u>transient</u> $\longleftrightarrow \rho_{xx} < 1 \longleftrightarrow \mathbb{E}_x N(x) < \infty$.

<u>Thm</u>: x recurrent, $\rho_{xy} > 0 \implies y$ recurrent, $\rho_{yx} = 1 = \rho_{xy}$.

Purpose: draw "accessibility graph":



x recurrent $\rightarrow$ all "downstream" states recurrent, all arrows among them bidirectional.

<u>Prop</u>: $\rho_{xy} > 0,\ \rho_{yz} > 0 \implies \rho_{xz} > 0$.

<u>Pf</u>: Strong Markov property, (Exercise) or find path w/ prob > 0.

<u>Def</u>: x recurrent $\rightarrow R_x := \{y : \rho_{xy} > 0\}$; $R := \{x \text{ recurrent}\}$.

<u>Cor</u>: (1) $\forall y \in R_x$, y recurrent and $\rho_{xy} = \rho_{yx} = 1$.

(2) $\exists x_1, x_2, \ldots \in R$ s.t. $R = R_{x_1} \sqcup R_{x_2} \sqcup \ldots$.

<u>Pf of Thm</u>: $\rho_{yx} = 1$: idea: if not, $\mathbb{P}_x[\text{never return}] > 0$.

$$\rho_{xy} > 0 \implies p^{(k)}(x,y) > 0 \text{ for some } k$$

$$p^{(k)}(x,y) = \sum_{y_1, \ldots, y_{k-1} \in S} p(x,y_1)\, p(y_1,y_2) \cdots p(y_{k-1},y) \quad \text{(CK equation)}$$

$$\implies \exists\, y_i \text{ s.t. } p(x,y_1), \ldots, p(y_{k-1},y) > 0.$$

Choose <u>minimal</u> such $k \rightsquigarrow y_i \neq x\ \forall i = 1, \ldots, k-1$.

$$0 = \mathbb{P}_x[\text{never return}] \geq \underbrace{p(x,y_1) \cdots p(y_{k-1},y)}_{>0}\ \underbrace{(1 - \rho_{yx})}_{=0} \quad \text{(Markov property)}.$$

<u>y recurrent</u>: idea: $\rho_{yx} = 1 \Rightarrow$ eventually visit x, x recurrent $\Rightarrow$ visit x infinitely many times, $\rho_{xy} > 0 \Rightarrow$ each time "chance" to go to y.

<u>Exercise</u>: Give argument using stopping times + Strong Markov property.

Our method: will show $E N(y) = \infty$.

$\rho_{yx} = 1 > 0 \Longrightarrow p^{(\ell)}(y,x) > 0$ for some $\ell \geq 1$.

$$\mathbb{E}_y N(y) = \mathbb{E}_y \sum_{n=1}^{\infty} \mathbb{1}\{Y_n = y\} = \sum_{n=1}^{\infty} p^{(n)}(y,y)$$

$$\geq \sum_{n\geq 1} \underbrace{p^{(\ell+n+k)}(y,y)}$$

$$= \sum_{v,w} p^{(\ell)}(y,v)\, p^{(n)}(v,w)\, p^{(k)}(w,y) \quad \text{(CK equation)}$$

$$\geq p^{(\ell)}(y,x)\, p^{(n)}(x,x)\, p^{(k)}(x,y)$$

$$\geq \underbrace{p^{(\ell)}(y,x)\, p^{(k)}(x,y)}_{>0} \underbrace{\sum_{n\geq 1} p^{(n)}(x,x)}_{+\infty} = +\infty.$$

<u>$\rho_{xy} = 1$</u>: reverse roles of x, y. $\blacksquare$

<u>Def</u>: Set of states $A \subseteq S$ is:

- <u>Closed</u> if $x \in A$, $\rho_{xy} > 0 \Longrightarrow y \in A$.
- <u>Irreducible</u> if $\rho_{xy} > 0 \;\forall\, x, y \in A$.

<u>Thm</u>: Let $A \subseteq S$ be <u>finite</u>.

(1) A closed $\Longrightarrow \exists\, x \in A$ recurrent.

(2) A closed, irreducible $\Longrightarrow$ all $x \in A$ recurrent.

<u>Pf</u>: (2) from (1) + Cor ($A = R_{x_0}$). (1): if all $x \in A$ transient,

$$\sum_{y\in A} \frac{\rho_{xy}}{1-\rho_{yy}} = \sum_{y\in A} \mathbb{E}_x N(y) = \mathbb{E}_x \sum_{y\in A} \sum_{n\geq 1} \mathbb{1}\{Y_n = y\} = \mathbb{E}_x \sum_{n\geq 1} 1 = \infty.$$

(3)

<u>Prop</u>: x recurrent $\to$ $R_x = \{y : \rho_{xy} > 0\}$ is closed + irreducible.

<u>Pf</u>: All $y \in R_x$ recurrent, have $\rho_{xy} = \rho_{yx} = 1$.

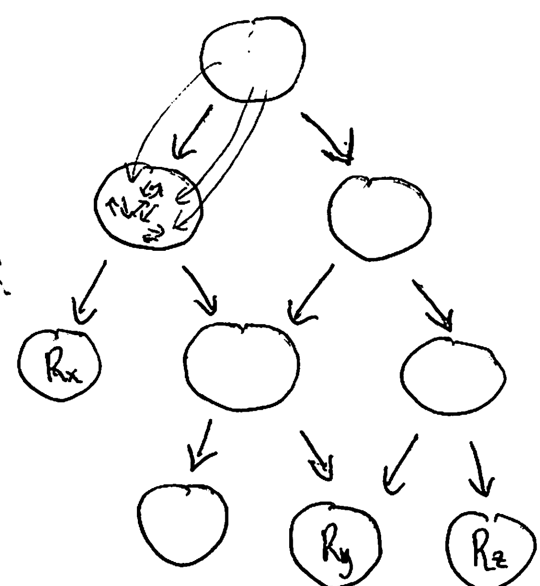
- <u>Closed</u>: $y \in R_x$, $\rho_{yz} > 0 \implies \rho_{xz} > 0 \implies z \in R_x$
- Irreducible: $y, z \in R_x \implies \rho_{yx} = \rho_{xz} = 1 \implies \rho_{yz} > 0$.

So, partition $R = R_{x_1} \sqcup \cdots \sqcup R_{x_n} \sqcup \cdots$ is into <u>closed</u>, <u>irred</u>, <u>disjoint</u>.

<u>Def</u>: For $x, y \in S$, write $x \leftrightarrow y$ if $\rho_{xy}, \rho_{yx} > 0$, or if $x = y$ (even if $\rho_{xx} = 0$.)

<u>Prop</u>: $\leftrightarrow$ is equiv. relation, and equivalence classes are irreducible subsets of S.

<u>Thm</u>: (Structure thm of countable MC) $\{C_i\}$ equiv classes of $\leftrightarrow$, $S = C_1 \sqcup C_2 \sqcup \cdots$. Draw (di-) graph on C_i, $C_i \to C_j$ if $\exists x \in C_i$, $y \in C_j$ w/ $\rho_{xy} > 0$. Then:



(1) This is a DAG (directed acyclic gph.)

(2) Every R_x is a leaf (some may be equal)

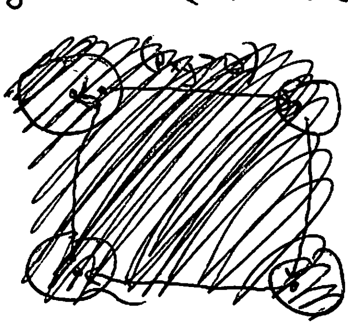
(3) Every leaf is closed; every closed C_i is leaf.

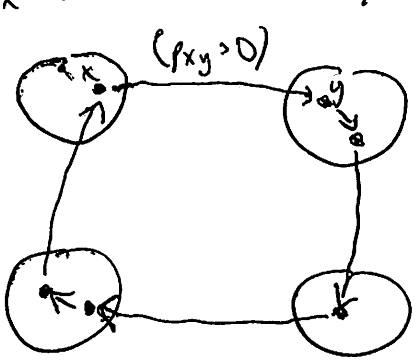
(4) Every <u>finite</u> leaf is some R_x.

(5) All recurrent states are in leaves, all other C_i contain only transient states.

<u>Pf</u>: (3) by defn. (2): every R_x is closed. (5) follows. (4) by Thm.

(1):





$\rho_{xy} > 0$; by transitivity of accessibility, $\rho_{yx} > 0$ as well.