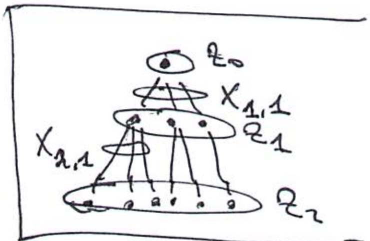
LECTURE 12: Branching processes: model of population growth,

each individual ⟶ iid. # of children (Galton-Watson model).

$X_{n,i} \overset{iid}{\sim} \mu$, $\in \mathbb{Z}_{\geq 0}$, $\mathbb{P}[X_{n,i} = k] = p_k$.

Process: $Z_0 := 1$, $Z_n := \sum_{i=1}^{Z_{n-1}} X_{n,i}$ for $n \geq 1$.



Q: How does long-term behavior depend on μ? (Cf. "replacement rate", etc.)

Important parameter: $m := \mathbb{E}[X_{n,i}] = \sum k p_k$, assume $< \infty$.

Rk: $\mathcal{F}_n := \sigma(X_{k,i} : 1 \leq k \leq n, i \geq 0)$ ⟶ (Z_n) adapted.

Investigating mgl properties:

$$\mathbb{E}[Z_{n+1} | \mathcal{F}_n] = \mathbb{E}\left[\sum_{i=1}^{Z_n} X_{n+1,i} \middle| \mathcal{F}_n\right] = \mathbb{E}\left[\sum_{i=1}^{\infty} X_{n+1,i} \mathbb{1}\{Z_n \geq i\} \middle| \mathcal{F}_n\right]$$

$$= \sum_{i=1}^{\infty} \mathbb{E}[X_{n+1,i} \underbrace{\mathbb{1}\{Z_n \geq i\}}_{\mathcal{F}_n\text{-m'able}} | \mathcal{F}_n]$$ (linearity of cond. exp., mon. conv. for cond. exp.)

$$= \sum_{i=1}^{\infty} \mathbb{1}\{Z_n \geq i\} \underbrace{\mathbb{E}[X_{n+1,i} | \mathcal{F}_n]}_{= m} = m Z_n.$$

Cor: $\mathbb{E} Z_n = m^n$, $\frac{Z_n}{m^n} =: M_n$ is mgl.

$M_n \geq 0$ ⟶ by Cor. of Doob MC, $M_n = \frac{Z_n}{m^n}$ converges a.s.

Rk: Power of general theory! No specific reasoning needed here.

Thm: (Subcritical case) If $m < 1$, then $Z_n \longrightarrow 0$ a.s.; further, $Z_n = 0$ $\forall$ suff large n, a.s., so $M_n \longrightarrow 0$ a.s. also.

Pf: $\mathbb{P}[Z_n \neq 0] = \mathbb{P}[Z_n \geq 1] \leq \mathbb{E} Z_n = m^n$. Borel-Cantelli ⟹ a.s., $Z_n \neq 0$ only finitely many times. ∎

Thm: (Critical case) Suppose $m = 1$. Then:

(1) If $p_1 = 1$, $Z_n = 1$ a.s. $\forall n$.

(2) If $p_1 < 1$, $Z_n = 0$ $\forall n$ suff. large a.s. (same as subcritical).

Pf: (1) $Z_{n+1} = \sum_{i=1}^{1} 1 = 1$.

(2) $M_n = Z_n \longrightarrow Z_\infty$ a.s. Integer-valued $\Rightarrow Z_n = Z_\infty$ $\forall n$ suff. large. a.s. $\to (k \geq 1)$

$$\mathbb{P}[Z_n = k \ \forall n \geq N_1] = \mathbb{P}[Z_{N_1} = Z_{N_1+1} = \cdots = k]$$

$$\leq \mathbb{P}[Z_{N_1} = Z_{N_1+1} = \cdots = Z_{N_2} = k]$$

$$= \underbrace{\mathbb{P}[Z_{N_1} = k]\,\mathbb{P}[Z_{N_1+1} = k \mid Z_{N_1} = k]}_{= \mathbb{P}[Z_{N_1} = Z_{N_1+1} = k],\ \text{etc.}} \cdots \mathbb{P}[Z_{N_2} = k \mid Z_{N_1} = \cdots = Z_{N_2-1} = k]$$

If $m = 1$, $p_1 < 1$, then $p_0 > 0$.

$\Longrightarrow \mathbb{P}[Z_{N_1+1} = k \mid Z_{N_1} = k] \leq 1 - p_0^k$, same for other probs.

$\Longrightarrow \mathbb{P}[Z_n = k \ \forall n \geq N_1] \leq (1 - p_0^k)^{N_2 - N_1} \ \forall N_2$

$\Longrightarrow$ taking $N_2 \uparrow \infty$, $\mathbb{P}[Z_n = k \ \forall n \geq N] = 0 \ \forall k \geq 1 \ \forall N$.

$\Longrightarrow \mathbb{P}[Z_\infty = k] = 0 \ \forall k \geq 1 \Longrightarrow Z_\infty = 0$ a.s. ∎

Supercritical case ($m > 1$): More complicated. First, let's consider

$\rho := \mathbb{P}[\text{"extinction"}] = \mathbb{P}[Z_n = 0 \text{ for some } n]$ ← growing sequence of events. ↓ increasing sequence.

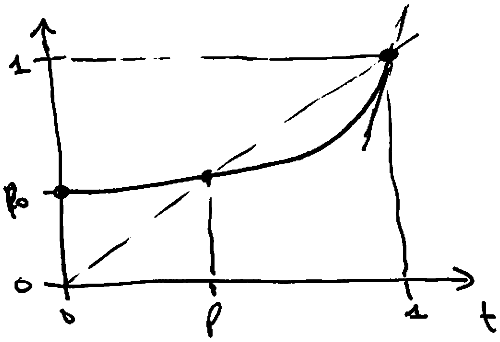
$= \mathbb{P}[Z_n = 0 \ \forall n \text{ suff. large}] = \lim_{n \uparrow \infty} \mathbb{P}[Z_n = 0]$.

<u>Thm</u>: Define prob. generating fn. $\phi(t) = \sum_{k\geq 0} p_k t^k = \mathbb{E}[t^{X_{1,1}}]$.

Then, (if $m>1$) ρ is the unique $\rho \in [0,1)$ s.t. $\phi(\rho) = \rho$. In particular, $0 \leq \rho \leq 1$.

<u>Pf</u>: $m > 1 \implies p_k > 0$ for some $k \geq 2$, so $p_0 < 1$ (obviously).

$\phi(0) = p_0$, $\phi(1) = 1$, $\phi'(t) = \sum_{k\geq 1} k p_k t^{k-1}$, $\phi''(t) = \sum_{k\geq 2} k(k-1) p_k t^{k-2}$ — both > 0 $\forall t > 0$.



$\phi'(1) = \sum k p_k = m > 1$.

$\implies$ fixed pt. exists.

ϕ strictly convex $\implies$ fixed pt. unique.

$\phi_1(t) := \phi(t) = \mathbb{E}[t^{Z_1}]$ ($Z_1 = X_{1,1}$), $\phi_n(t) := \mathbb{E}[t^{Z_n}]$. Recursion:

$$\phi_{n+1}(t) = \mathbb{E}\left[\mathbb{E}\left[t^{Z_{n+1}} \mid \mathcal{F}_n\right]\right] = \mathbb{E}\left[\mathbb{E}\left[t^{X_{n+1,1} + \dots + X_{n+1,Z_n}} \mid \mathcal{F}_n\right]\right]$$

Exercise: verify.

$$\overset{\downarrow}{=} \mathbb{E}\left[\left(\mathbb{E}\, t^{X_{n+1,1}}\right) \cdots \left(\mathbb{E}\, t^{X_{n+1,Z_n}}\right)\right] = \mathbb{E}\,\phi(t)^{Z_n} = \phi_n(\phi(t))$$

$\implies$ Inductively, $\phi_n(t) = (\underbrace{\phi \circ \cdots \circ \phi}_{n})(t)$.

$\rho_n := \mathbb{P}[Z_n = 0] = \phi_n(0)$, $\rho_{n+1} = \phi_{n+1}(0) = \phi(\rho_n)$.

$\rho = \lim_{n\to\infty} \rho_n$ coincides w/ unique fixed pt. (consider picture.)

<u>Cor</u>: $\rho = 0$ (i.e., population a.s. survives forever) $\iff p_0 = 0$.

<u>Pf</u>: Either $m = 1$, $p_1 = 1$ (critical) or $m > 1$ and $\phi(0) = p_0 = 0$.

<u>Cor</u>: If $m > 1$, then cannot have $Z_n \longrightarrow 0$ a.s.

<u>Q</u>: How fast does Z_n grow? Does $M_n = \frac{Z_n}{m^n} \longrightarrow 0$?

<u>Rk</u>: If $M_n \xrightarrow{L_1} M_\infty$, then $1 = \mathbb{E}\,M_n \longrightarrow \mathbb{E}M_\infty = 1$, so $M_\infty > 0$ w/ positive probability, and on this event $Z_n \simeq m^n$, grows <u>exponentially</u>.