LECTURE 11

Bit of useful notation:

<u>Def</u>: For X r.v., $p \geq 1$, $\|X\|_p := \left(\mathbb{E}|X|^p\right)^{1/p}$, $L^p := \{X : \|X\|_p < \infty\}$.

Can restate last time's main result as:

<u>Thm</u>: (DMC) If (M_n) $\left\{\begin{matrix} \text{mgl} \\ \text{sub} \\ \text{super} \end{matrix}\right\}$, $\|M_n\|_1$ bounded, then $\exists M_\infty \in L^1$ s.t. $M_n \xrightarrow{a.s.} M_\infty$.

$\Rightarrow \mathbb{E}M_n \to \mathbb{E}M_\infty$.

<u>Q</u>: When does $M_n \xrightarrow{L^1} M_\infty$, i.e. $\|M_n - M_\infty\|_1 \to 0$?

<u>Ex</u>: S_n SRW, $T := \{\min n : S_n = a\}$, $M_n := S_{T \wedge n}$ $\to$ NO.

<u>Ex</u>: $X_i = \left\{\begin{matrix} 0 & \text{w/p } 1/2 \\ 2 & \text{w/p } 1/2 \end{matrix}\right.$, $M_n := \prod_{i=1}^n X_i$ $\longrightarrow$ NO.

<u>Def</u>: R.v. (Y_i) are <u>uniformly integrable (UI)</u> if $\lim_{t\to\infty} \sup_i \mathbb{E}\left[|Y_i| \mathbb{1}\{|Y_i| \geq t\}\right] = 0$.

<u>Rk</u>: (1) Similar flavor to Lindeberg CLT condition.

(2) (Y_i) UI $\Rightarrow$ $\|Y_i\|_1$ bounded.

(3) $|Y_i| \leq Z \in L^1 \Rightarrow (Y_i)$ UI $\leftarrow$ <u>weaker version</u> of being dominated a la dom. conv.

<u>Thm</u>: $Y_i \xrightarrow{L^1} Y_\infty \iff Y_i \xrightarrow{\mathbb{P}} Y_\infty$ and Y_i UI

$\Downarrow$ $\Uparrow$ $\Uparrow$

<u>Dom. conv.</u>: $\mathbb{E}Y_i \to \mathbb{E}Y_\infty$ $Y_i \xrightarrow{a.s.} Y_\infty$ $|Y_i| \leq Z \in L^1$

<u>Cor</u>: (L^1 MC) If (M_n) UI $\left\{\begin{matrix} \text{mgl} \\ \text{sub} \\ \text{super} \end{matrix}\right\}$, then $M_n \xrightarrow{a.s., L^1} M_\infty \in L^1$.

<u>Ex</u>: Product martingale $M_n = \left\{\begin{matrix} 2^n & \text{w/p } \frac{1}{2^n} \\ 0 & \text{w/p } 1 - \frac{1}{2^n} \end{matrix}\right.$ $\to \sup_n \mathbb{E}\left[|M_n| \mathbb{1}\{|M_n| \geq t\}\right] = 2^n \cdot \frac{1}{2^n} = 1 \neq 0$.

(2) An "L^p-ish" way to check UI:

<u>Thm</u>: Suppose $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ has $\frac{f(x)}{x} \longrightarrow \infty$. If $\mathbb{E} f(Y_i)$ bounded, then (Y_i) are UI.

<u>Pf</u>: $$\mathbb{E}\left[|Y_i| \mathbb{1}\{|Y_i| \geq t\}\right] \leq \underbrace{\sup\left\{\frac{x}{f(x)} : x \geq t\right\}}_{\to 0.} \underbrace{\mathbb{E}\left[f(|Y_i|)\right]}_{\text{bounded}}$$

<u>Ex</u>: $f(x) = x^p$ for $p > 1$ $\rightsquigarrow$ enough to have $\|Y_i\|_p$ bdd.

$f(x) = x \log(1+x)$.

<u>Lem</u>: $(p=2)$ (M_n) mgl has $\|M_n\|_2$ bdd $\iff$ $M_0 \in L^2$, $\sum_{i=1}^{\infty} \underbrace{\mathbb{E}(M_i - M_{i-1})^2}_{= \|M_i - M_{i-1}\|_2^2} < \infty$

<u>Pf</u>: $\mathbb{E}(M_d - M_c)(M_b - M_a) = 0$

$\forall d > c \geq b > a$; use $M_n = M_0 + \sum_{i=1}^{n} (M_i - M_{i-1})$.

<u>Maximal inequalities</u> (towards L^p convergence)

<u>Prop</u>: (M_n) <u>sub</u>-mgl, T stopping time $\Longrightarrow$ $\mathbb{E} M_0 \leq \mathbb{E} M_{T \wedge n} \leq \mathbb{E} M_n$.

<u>Pf</u>: $H_n := \mathbb{1}\{T < n\}$ $\to$ $(H \cdot M)$ submgl, $(H \cdot M)_n = M_n - M_{T \wedge n}$

$(H \cdot M)_0 = 0$.

<u>Thm</u>: (Doob maximal ineq) (M_n) submgl, $t > 0$, $\mathrm{MAX}_n := \max_{0 \leq k \leq n} M_k$

$$\mathbb{P}[\mathrm{MAX}_n \geq t] \leq \frac{\mathbb{E}[M_n^+ \mathbb{1}\{\mathrm{MAX}_n \geq t\}]}{t} \leq \frac{\mathbb{E} M_n^+}{t}$$

<u>"uniform Markov"</u>

<u>Pf</u>: $T := \min\{n : M_n \geq t\}$. Prop $\Longrightarrow$ $\mathbb{E} M_{T \wedge n} \leq \mathbb{E} M_n$.

$A := \{\mathrm{MAX}_n \geq t\} = \{T \leq n\}$

$$\to \underbrace{\mathbb{E} M_{T \wedge n} \mathbb{1}\{A\}}_{\geq t \mathbb{P}[A]} + \cancel{\mathbb{E} \underset{\downarrow \atop M_n}{M_{T \wedge n}} \mathbb{1}\{A^c\}} \leq \mathbb{E} M_n \mathbb{1}\{A\} + \cancel{\mathbb{E} M_n \mathbb{1}\{A^c\}}.$$

Appl: X_i ind., $\mathbb{E}X_i = 0$, $S_n = \sum_{i=1}^n X_i \rightsquigarrow M_n := \exp(\lambda S_n)$ submgl.

$$\longrightarrow \mathbb{P}\left[\max_{k=1}^{n} S_k \ge t\right] = \mathbb{P}\left[\max_{k=1}^{n} \exp(\lambda S_k) \ge \exp(\lambda t)\right] \le \frac{\mathbb{E}\exp(\lambda S_n)}{\exp(\lambda t)}$$

$\Longrightarrow$ Hoeffding-type ineq. automatically true for maximum!

Thm: (L^p maximal ineq) (M_n) submgl, $M_n \ge 0$, $p > 1$

$$\Longrightarrow \|MAX_n\|_p \le \frac{p}{p-1}\|M_n\|_p.$$

Rk: $M_n \ge 0$ is WLOG, since M_n submgl $\rightarrow M_n^+$ submgl.

Cor: (L^p MC) (M_n) $\boxed{\text{mgl}}$, $p > 1$, $\|M_n\|_p$ bounded, then $M_n \xrightarrow{a.s., L^p} M_\infty \in L^p$. !

Pf: $\|M_n\|_1 \le \|M_n\|_p$ bdd $\Longrightarrow$ by DMC, $M_n \xrightarrow{a.s.} M_\infty \in L^1$.

$|M_n|$ submgl. L^p maximal $\Longrightarrow \mathbb{E}\left(\max_{k=0}^{n} |M_k|\right)^p \lesssim \|M_n\|_p^p$ bounded.

Monotone conv $\Longrightarrow \mathbb{E}\sup |M_n|^p < \infty$.

$|M_n - M_\infty|^p \le \left(2\sup_n |M_n|\right)^p \in L^1$. Dom. conv. $\Longrightarrow \mathbb{E}|M_n - M_\infty|^p \to 0$
$\qquad = \|M_n - M_\infty\|_p^p$.

Pf: (of Thm) Threshold at some $C > 0$:

$$\mathbb{E}(MAX_n \wedge C)^p = \mathbb{E}\left[p\int_0^{MAX_n \wedge C} t^{p-1}\,dt\right] = \mathbb{E}\left[p\int_0^\infty t^{p-1}\mathbb{1}\{MAX_n \wedge C \ge t\}\,dt\right]$$

$$\overset{\text{(Fubini)}}{=} \int_0^\infty p t^{p-1}\mathbb{P}[MAX_n \wedge C \ge t]\,dt = \int_0^C p t^{p-1}\mathbb{P}[MAX_n \ge t]\,dt$$

$$\overset{\text{(Doob maximal)}}{\le} \int_0^C p t^{p-2}\mathbb{E}\left[M_n \mathbb{1}\{MAX_n \ge t\}\right]dt$$

$$= \int_0^\infty p t^{p-2}\mathbb{E}\left[M_n \mathbb{1}\{MAX_n \wedge C \ge t\}\right]dt$$

④ $\overset{\text{(Fubini)}}{=} \mathbb{E}\left[M_n \cdot p \cdot \int_0^\infty t^{p-2} \mathbb{1}\{\mathrm{MAX}_n \wedge C \geq t\}\, dt\right]$

$= \mathbb{E}\left[M_n \cdot p \cdot \int_0^{\mathrm{MAX}_n \wedge C} t^{p-2}\, dt\right]$

$= \mathbb{E}\left[M_n \cdot p \cdot \frac{1}{p-1} (\mathrm{MAX}_n \wedge C)^{p-1}\right]$

$\overset{\text{(Hölder)}}{\leq} \frac{p}{p-1} \|M_n\|_p \left(\mathbb{E}\left[(\mathrm{MAX}_n \wedge C)^{p-1}\right]\right)^{\frac{p-1}{p}}$

Rearrange, take $C \uparrow \infty$.

<u>Applications</u>:

<u>Thm</u>: (Kolmogorov 2 series) X_i ind. $\in L^2$ s.t. $\sum |\mathbb{E}X_i|, \sum \mathrm{Var}\, X_i < \infty$.
~~WLOG $\mathbb{E}X_1 = 0$~~ $\Rightarrow \sum X_i$ converges a.s. and in L^2.

<u>Pf</u>: WLOG $\mathbb{E}X_i = 0$. Then, $S_n := \sum_{i=1}^n X_i$ is mgl <u>bdd. in L^2</u>
$\rightsquigarrow$ follows by L^2MC.

By earlier remark on increments.

<u>Ex</u>: $\sum_{n=1}^\infty \pm \frac{1}{n^{\frac{1}{2}+\varepsilon}}$ converges a.s.

<u>Ex</u>: $\sum_{n=0}^\infty \frac{g_n}{\sqrt{n!}} z^n = f(z)$ for $g_n \overset{iid}{\sim} N(0,1)$
well-defined, <u>Gaussian analytic fn.</u>
L^pMC $\Rightarrow$ converges in L^p for all p.
Justifies computing moments $\Rightarrow$ Law$(f(z)) = N(0, \exp(z^2))$, etc.

<u>Rk</u>: Appl to RW, but the proof uses martingale theory via L^p maximal ineq. on $|S_n|$.