Stopping Times + Optional Stopping

Recall martingale betting strategy: can view as $\tilde{M}_n := \sum_{i=2}^{n} 2^{i-2} \cdot \underbrace{X_i}_{\text{Uniform}\{\pm 1\}}$ $\quad M_n = \tilde{M}_n$ "stopped" at first time that $X_n = +1$.

$T := \min\{n : X_n = +1\} \quad \longrightarrow \quad M_n = \tilde{M}_{T \wedge n} = \begin{cases} \tilde{M}_n & : n \le T \\ \tilde{M}_T & : n \ge T \end{cases}$

Def: $a \wedge b := \min\{a, b\}$, $a \vee b := \max\{a, b\}$

$\tilde{M}$, M both martingales.

Def: $(\mathcal{F}_n)$ filtration, $T : \Omega \to \{0, 1, 2, \dots, \infty\}$ is *stopping time* if $\{\omega : T(\omega) = n\} \in \mathcal{F}_n \ \forall n \quad \Longleftrightarrow \quad \{\omega : T(\omega) \le n\} \in \mathcal{F}_n \ \forall n$

"with info in $\mathcal{F}_n$, can decide if T has happened by time n."

Ex: S_n = SRW $\rightsquigarrow$ stopping times $T = \min\{n : S_n = a\}$ ("hitting time")

$T = \min\{n : S_n \in \{-a, b\}\}$ ("exit time")

Thm: If T stopping time, (M_n) $\begin{cases} \text{mgl} \\ \text{submgl} \\ \text{supermgl} \end{cases}$ then $(M_{T \wedge n})$ also $\begin{cases} \text{mgl} \\ \text{submgl} \\ \text{supermgl} \end{cases}$.

Pf: $H_n := \mathbb{1}\{T \ge n\} = 1 - \mathbb{1}\{T \le n-1\} \rightsquigarrow H_n$ is predictable, $\boxed{\ge 0}$.

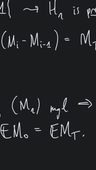
$(H \cdot M)_n = \sum_{i=2}^{n} H_i (M_i - M_{i-1}) = \sum_{i=2}^{T \wedge n} (M_i - M_{i-1}) = M_{T \wedge n} - M_0$.

$M_0 + (H \cdot M)_n = M_{T \wedge n}$. ∎

Cor: T bdd. stopping time ($T < t$ a.s.), (M_n) mgl $\Rightarrow \mathbb{E} M_T = \mathbb{E} M_0$.

Pf: $(M_{T \wedge n})_n$ also mgl $\Rightarrow \mathbb{E} M_{T \wedge t} = \mathbb{E} M_0 = \mathbb{E} M_T$.

Q: Does this hold for more general T?



Ex: (S_n) SRW, $T := \min\{n : S_n = 1\}$

Can show $T < \infty$ a.s., but

$\mathbb{E} S_T = 1 \neq 0 = \mathbb{E} S_0$, while $\mathbb{E} S_{T \wedge n} = 0$.

[**Preliminary Q:** M_T is r.v. (=$\mathcal{F}$-mble) — what σ-algebra? $\mathcal{M}(T(\Omega))$.

Def: (stopped σ-algebra) For $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_n \subseteq \cdots \subseteq \mathcal{F}$ filtration, T stopping time $\rightsquigarrow \mathcal{F}_T := \{A \in \mathcal{F} : A \cap \{\omega : T(\omega) \le n\} \in \mathcal{F}_n \ \forall n \ge 0\}$.

Intuition: $\mathcal{F}_n$ = info up to time n, $\mathcal{F}_T$ = info up to random time T.

i.e. A s.t. "if T happened, we can decide if A happened."

Prop: T is $\mathcal{F}_T$-mble; (M_n) adapted $\Rightarrow M_T$ is $\mathcal{F}_T$-mble.]

Thm: (Doob Optional Stopping Thm) (M_n) mgl, T stopping time. Suppose:

(1) $T \le t$ a.s. for some $t \overset{\text{const}}{<} \infty$ — OR — ("bdd in time")

(2) $T < \infty$ a.s. and $|M_n| \le C \overset{\text{const}}{} \ \forall n$ a.s. — OR — ("bdd in space")

(3) $\mathbb{E} T < \infty$ and $|M_n - M_{n-1}| \le C$ a.s. $\forall n$. (in between)

Then $\mathbb{E}|M_T| < \infty$, and $\mathbb{E} M_T = \mathbb{E} M_0$.

Pf: (1): Corollary ($M_{T \wedge n}$ mgl, take $n = t$ $\rightsquigarrow \mathbb{E} M_{T \wedge t} = \mathbb{E} M_0$.

(2): $T < \infty$ a.s. $\Rightarrow M_{T \wedge n} \xrightarrow{n \to \infty} M_T$ a.s., all bounded uniformly by C.

bounded conv. $\Rightarrow \mathbb{E} M_{T \wedge n} \longrightarrow \mathbb{E} M_T \Rightarrow \mathbb{E} M_T = \mathbb{E} M_0$.

(3): $|M_{T \wedge n} - M_0| = \left| \sum_{i=2}^{T \wedge n} (M_i - M_{i-1}) \right| \le C \cdot (T \wedge n) \le CT$ ($\mathbb{E} M_0$)

$\mathbb{E} T < \infty$; $M_{T \wedge n} - M_0 \xrightarrow[n \to \infty]{} M_T - M_0$ a.s.

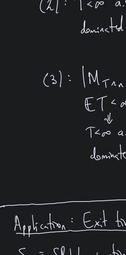
$T < \infty$ a.s.

dominated conv. $\Rightarrow \mathbb{E}[M_{T \wedge n} - M_0] \to \mathbb{E}[M_T - M_0] = \mathbb{E} M_T - \mathbb{E} M_0$

$\mathbb{E} M_{T \wedge n} - \mathbb{E} M_0 = 0$ ($= 0$)

Application: Exit times in SRW $\quad (a, b > 0)$

S_n = SRW, martingale. $T := \min\{n : S_n \in \{-a, b\}\}$



Q: Distribution of T, S_T? Joint distribution of (T, S_T)?

Plan: Design martingales based on S_n, use OST on them (with T).

T not bounded a.s., $|S_n|$ not bounded a.s.

$\Rightarrow$ Can only use OST (3). (Note: $|S_n - S_{n-1}| = 1$ bdd.)

Lem: $\mathbb{E} T < \infty$.

Pf: $S_n = \sum_{i=2}^{n} X_i \overset{\text{Unif}\{\pm 1\}}{}$ iid. If $X_{k+2} = \cdots = X_{k+(a+b)} = +1 \Rightarrow T \le k + (a+b)$.

$Y_i := \mathbb{1}\{i^{\text{th}}$ string of $(a+b)$ of X_j's $= +1\}$, $\quad G := \max\{i : Y_i = 1\}$.

X_j's: $\underbrace{+1 \ -1 \ +2}_{Y_2 = 0} \quad \underbrace{-1 \ -1 \ -1}_{Y_3 = 0} \quad \underbrace{+1 \ +1 \ +1}_{Y_4 = 1}$ $\quad (+1+b=3)$

$\leftarrow G = 3$, $T \le 9$

Y_i are iid $\text{Ber}(2^{-(a+b)})$, $\quad$ Law$(G) \sim \text{Geom}(2^{-(a+b)})$

$T \le (a+b) G \quad \Longrightarrow \quad \mathbb{E} T \le (a+b) 2^{a+b}$. ←

Def: $G \sim \text{Geom}(p)$ has $\mathbb{P}[G = k] = (1-p)^{k-1} \cdot p$. $\quad \rightsquigarrow$ **Prop:** $\mathbb{E} G = \frac{1}{p}$.

- **Martingale 1:** $M_n = S_n$

OST (3) $\Rightarrow \mathbb{E} S_T = \mathbb{E} S_0 = 0 = p \cdot b + (1-p)(-a) = (a+b)p - a$

$\Rightarrow p = \frac{a}{a+b}$

$p := \mathbb{P}[\text{"exit up"}] = \mathbb{P}[S_T = \{b\}]$, $\quad 1-p = \mathbb{P}[S_T = -a]$

Cor: $\mathbb{P}[\text{exit up (at } b)] = \frac{a}{a+b}$, $\quad \mathbb{P}[\text{exit down (at } -a)] = \frac{b}{a+b}$.

↳ characterizes Law(S_T). $\quad (\mathcal{F}_n = \sigma(X_2, \dots, X_n))$

- **Martingale 2:** $M_n = S_n^2 - n$ is martingale.

Pf: $\mathbb{E}\left[\underbrace{S_n^2 - n}_{M_n} \middle| \mathcal{F}_{n-1}\right] = \mathbb{E}\left[(S_{n-1} + X_n)^2 - n \middle| \mathcal{F}_{n-1}\right]$

$= \mathbb{E}\left[S_{n-1}^2 + 2 S_{n-1} X_n + X_n^2 - n \middle| \mathcal{F}_{n-1}\right]$

$= S_{n-1}^2 + 2 S_{n-1} \cancel{\mathbb{E}[X_n | \mathcal{F}_{n-1}]} + \mathbb{E}[X_n^2 | \mathcal{F}_{n-1}] - n$ ($\mathbb{E}[X_n] = 0$, $+1$, $= \mathbb{E}[X_n^2] = 1$)

$= S_{n-1}^2 - (n-1) = M_{n-1}$.

Want to use OST (3). Would need $|M_n - M_{n-1}| = |2 S_{n-1} X_n + 1|$ bdd a.s., unif in n.

Instead use OST on $(M_{T \wedge n}) \leftarrow$ still mgl.

$|M_{T \wedge n} - M_{T \wedge (n-1)}| \le 2 a \vee b$, bdd. a.s.

$\le a \vee b \quad \le a \vee b$

$S_T = \begin{cases} -a & \text{w.l. prob } \frac{b}{a+b} \\ b & \text{w.l. prob } \frac{a}{a+b} \end{cases}$

OST $\Rightarrow \mathbb{E} M_{T \wedge T} \ (= \mathbb{E} M_T) = \mathbb{E} M_{T \wedge 0} = \mathbb{E} M_0 = 0$.

$\Longrightarrow 0 = \mathbb{E} M_T = \mathbb{E}[S_T^2 - T]$

$\Longrightarrow \mathbb{E} T = \mathbb{E} S_T^2 = a^2 \cdot \frac{b}{a+b} + b^2 \cdot \frac{a}{a+b} = \frac{ab(a+b)}{a+b}$

Cor: $\mathbb{E} T = \mathbb{E}[\text{time to exit } (-a, b)] = ab$.

Further sequence of "polynomial martingales" giving more and more info about Law(T), Law(S_T), Law$((T, S_T))$. Analogy for $\mathbb{E}[T \mathbb{1}\{S_T = b\}]$.

(!!) Can fully determine joint law like this! $\quad \leftrightarrow$ for $\mathbb{E}[T \mid S_T]$.

Prop: $M_n := S_n^3 - 3 n S_n$ is a martingale and OST can be applied at time T.

$\Rightarrow \mathbb{E} M_T = \mathbb{E} S_T^3 - 3 \mathbb{E} T S_T - \mathbb{E} M_0 = 0$.

Before: $\mathbb{E} T \mathbb{E} S_T$ (can compute!)

Now: $\mathbb{E}[T S_T]$ (not usually)

$\rightsquigarrow$ formula for $\mathbb{E}[T S_T]$ also $\mathbb{E} T = -a \, \mathbb{E}[T \mathbb{1}\{S_T = -a\}] + b \, \mathbb{E}[T \mathbb{1}\{S_T = b\}]$