LECTURE 7

End of pf. of cond. expectation existence: we did case $X \geq 0$.
In general, $X^+ := X \mathbb{1}\{X \geq 0\}$, $X^- := -X \mathbb{1}\{X < 0\}$, so that $X = X^+ - X^-$. Then, can check correct definition is:

$$\mathbb{E}[X|\mathcal{G}] := \mathbb{E}[X^+|\mathcal{G}] - \mathbb{E}[X^-|\mathcal{G}]$$

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Main properties of conditional expectation:

- Linearity: $\mathbb{E}[aX + bY|\mathcal{G}] = a\,\mathbb{E}[X|\mathcal{G}] + b\,\mathbb{E}[Y|\mathcal{G}]$

- Monotonicity: $X \leq Y$ a.s. $\Longrightarrow \mathbb{E}[X|\mathcal{G}] \leq \mathbb{E}[Y|\mathcal{G}]$ a.s.
  Pf: WLOG $X = 0$. $E := \mathbb{E}[Y|\mathcal{G}] \rightsquigarrow \mathbb{E}[E\mathbb{1}\{E<0\}] = \mathbb{E}[Y\mathbb{1}\{E<0\}] \geq 0$.

- Factorization: $\mathbb{E}|X|, \mathbb{E}|Y|, \mathbb{E}|XY| < \infty$ and X $\mathcal{G}$-m'ble $\Longrightarrow$ $\mathbb{E}[XY|\mathcal{G}] = X\,\mathbb{E}[Y|\mathcal{G}]$.
  Pf: Say $X = \mathbb{1}_A$ for $A \in \mathcal{G}$: for $B \in \mathcal{G}$,
  $$\mathbb{E}[X\,\mathbb{E}[Y|\mathcal{G}] \cdot \mathbb{1}_B] = \mathbb{E}[\mathbb{E}[Y|\mathcal{G}]\,\mathbb{1}_{A \cap B}] = \mathbb{E}\,Y\mathbb{1}_{A\cap B}$$
  $$= \mathbb{E}\,XY\mathbb{1}_B \rightsquigarrow \text{satisfies defining property.}$$
  Then, standard argument: indicators $\to$ step functions (linearity) $\rightsquigarrow$ m'ble functions (monotone conv.)

- Independence: If X ind. of $\mathcal{G}$ (i.e., any $A \in \sigma(X)$, $B \in \mathcal{G}$ ind.) then $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X]$. In particular, X, Y ind. $\Rightarrow \mathbb{E}[X|Y] = \mathbb{E}[X]$.
  Pf: $\hookrightarrow =: E$. Then, $\mathbb{E}[E\mathbb{1}_B] = \mathbb{E}[X\mathbb{1}_B] = \mathbb{E}[X]\,\mathbb{E}\mathbb{1}_B$
  $$= \mathbb{E}[\mathbb{E}[X] \cdot \mathbb{1}_B].$$

- Tower rule: $\mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \mathcal{F} \Rightarrow \mathbb{E}[\mathbb{E}[X|\mathcal{G}_2]|\mathcal{G}_1] = \mathbb{E}[X|\mathcal{G}_1]$
In particular, w/ $\mathcal{G}_1 = \{\emptyset, \Omega\}$, $\mathbb{E}[\mathbb{E}[X|\mathcal{G}]] = \mathbb{E}[X]$.

$$\mathbb{E}[\mathbb{E}[X|\mathcal{G}_1]\mathbb{1}_A] = \mathbb{E}[X\mathbb{1}_A] = \mathbb{E}[\mathbb{E}[X|\mathcal{G}_2]\mathbb{1}_A]$$

$A \in \mathcal{G}_1$, also $\mathcal{G}_2$

- Jensen ineq: f convex, $\mathbb{E}|X|, \mathbb{E}|f(X)| < \infty \Rightarrow$

$$\mathbb{E}[f(X)|\mathcal{G}] \geq f(\mathbb{E}[X|\mathcal{G}]) \quad \text{a.s.}$$ both sides random!

Martingales (discrete time):

Def: $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_{n-1} \subseteq \mathcal{F}_n \subseteq \mathcal{F}$ is filtration.
(Intuition: $\mathcal{F}_i$ = "info available at time i.")
$(X_n)_{n\geq 0}$ r.v.'s are adapted to filtration if X_i is $\mathcal{F}_i$ m'ble, $\forall i$.
$(X_n)_{n\geq 1}$ are predictable/previsible if X_i is $\mathcal{F}_{i-1}$ m'ble, $\forall i \geq 1$.

Def: $(M_n)_{n\geq 0}$ is $\begin{Bmatrix}\text{martingale}\\ \text{submartingale}\\ \text{supermartingale}\end{Bmatrix}$ if (1) $\mathbb{E}|M_n| < \infty$, (2) adapted,

and (3) $\mathbb{E}[M_{n+1} | \mathcal{F}_n] \begin{Bmatrix} = \\ \geq \\ \leq \end{Bmatrix} M_n$ a.s., $\forall n \geq 0$

Rk: Same properties hold w.r.t. filtration $\tilde{\mathcal{F}}_n = \sigma(M_0, M_1, \ldots, M_n)$.

Prop: For $\begin{Bmatrix}\text{mgl}\\ \text{submgl}\\ \text{super mgl}\end{Bmatrix}$, $\mathbb{E}M_n \begin{Bmatrix} = \\ \geq \\ \leq \end{Bmatrix} \mathbb{E}M_0 \;\; \forall n \geq 1$. Pf: Tower property + induction.

Prop: $\mathbb{E}[M_{n+k} | \mathcal{F}_n] = M_n \;\; \forall n \geq 0, k \geq 1$. Pf: Same.

Examples:

or just independent

(1) Random walk: X_i $\boxed{\text{iid}}$, $\mathbb{E} X_i = 0$, $S_0 := 0$, $S_n := \sum_{i=1}^{n} X_i$.
Then (S_n) is martingale (w.r.t. $\mathcal{F}_n = \sigma(X_1, \dots, X_n) = \sigma(S_1, \dots, S_n)$):

$$\mathbb{E}[S_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[S_n + X_{n+1} \mid \mathcal{F}_n] = \underbrace{\mathbb{E}[S_n \mid \mathcal{F}_n]}_{= S_n \text{ since } \mathcal{F}_n \text{ m'ble.}} + \underbrace{\mathbb{E}[X_{n+1} \mid \mathcal{F}_n]}_{= 0 \text{ by ind. property.}}$$

Think: LLN, CLT, etc. for mgl's?

[$\Delta_n := M_n - M_{n-1}$ martingale differences/increments
$M_n = M_0 + \sum_{i=1}^{n} \Delta_i$ mgl $\binom{\text{sub-mgl}}{\text{super-mgl}}$ iff $\mathbb{E}[\Delta_{n+1} \mid \mathcal{F}_n] = 0$ $\binom{\geq}{\leq}$ $\forall n$.]

(2) Random product/geometric RW: X_i iid, $X_i \geq 0$ a.s., $\mathbb{E} X_i = 1$
$\longrightarrow$ $M_0 = 1$, $M_n = \prod_{i=1}^{n} X_i$ is mgl.

Ex: $X_i = \begin{cases} 0 & \text{w/prob } 1/2 \\ 2 & \text{w/prob } 1/2 \end{cases}$. Exc: $M_n \longrightarrow 0$ a.s.
(Note $0 \neq \mathbb{E} M_n = 1$!)

Very non-RW behavior!

(3) Doob martingale: X any r.v. w/ $\mathbb{E}|X| < \infty$, $(\mathcal{F}_n)$ any filtration
$\longrightarrow$ $M_n := \mathbb{E}[X \mid \mathcal{F}_n]$ is mgl. (tower rule)

(Say $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_N = \mathcal{F}$, then $M_0 = \mathbb{E} X$,
$M_N = X$, in between "gradually reveal info", "forecast X".)

(4) Martingale transform: (H_n) predictable, (M_n) martingale $\longrightarrow$

$$(H \cdot M)_n = \begin{cases} 0 & \text{if } n = 0 \\ \sum_{i=1}^{n} H_i \underbrace{(M_i - M_{i-1})}_{= \Delta_i} & \text{if } n \geq 1 \end{cases} \quad \text{is mgl.}$$

Also, if $H \geq 0$ and (M_n) sub/super-mgl, then $H \cdot M$ sub/super-mgl.

(4)

(Increments of $H \cdot M$ are $\tilde{\Delta}_n = H_n \Delta_n$, $E[\tilde{\Delta}_{n+1} | \mathcal{F}_n] = H_{n+1} E[\Delta_{n+1} | \mathcal{F}_n] = 0$.)

Rk: Precursor to stochastic integral, $\int_0^t H \, dM$ (cf. Stieltjes integral.)

Ex: Mgl transform of $\underbrace{\text{simple RW}}_{\text{SRW}}$: $\overset{\Delta_i}{\overset{\shortparallel}{X_i}} = \begin{cases} +1 & \text{w/prob } 1/2 \\ -1 & \text{w/prob } 1/2 \end{cases}$, $\overset{M_n}{\overset{\shortparallel}{S_n}} = \sum_1^n X_i$,
$\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$, (H_n) previsible, i.e. $H_n \in \mathcal{F}_{n-1}$,
i.e. $H_n = h_n(X_1, \ldots, X_{n-1})$.

$$(H \cdot S)_n = \sum_{i=1}^n H_i X_i = \sum_{i=1}^n h_i(X_1, \ldots, X_{i-1}) X_i.$$

Gambling interpretation: X_i = outcome of ith bet,
H_i = size of ith bet, (can depend on history!)
$(H \cdot S)_n$ = profit after n bets

Martingale: Name for specific strategy

$$H_1 = 1, \quad H_n = \begin{cases} 2H_{n-1} & \text{if } X_{n-1} = -1 \\ 0 & \text{if } X_{n-1} = +1 \end{cases}$$

I.e., double my bet until I win, then stop.

Analysis: a.s. some $X_i = 1$, let n = first such time.
$H_1 = 1 = 2^0$, $H_2 = 2^1$, $H_3 = 2^2$, ..., $H_n = 2^{n-1}$, $H_{n+1} = 0 = H_{n+k}$.
$X_1 = -1$, $X_2 = -1$, $X_3 = -1$, ..., $X_n = +1$
profit at all times $(\geq n) = 2^{n-1} - 2^{n-2} - \cdots - 2 - 1 = 1$.

Always win?!

Rk:
- RW: martingale not converging
- GRW: mgl converging below mean ($EM_n = 1$, $M_n \to 0$)
- $(H \cdot S)$: mgl converging above mean ($EM_n = 0$, $M_n \to 1$)