Assignment 5

Probability Theory II
(EN.553.721, Spring 2025)

Assigned: April 17, 2025 Due: 11:59pm EST, April 29, 2025

Solve any three out of the four problems. If you solve more, we will grade the first four solutions you include. Each problem is worth an equal amount towards your grade.

Submit solutions in LaTeX. Write in complete sentences. Include and justify all steps of your arguments, but avoid writing excessive explanation that is not contributing to your solution.

Keep in mind the late submission policy: you may use a total of five late days for homework submissions over the course of the semester without penalty. If you need an extension beyond these, *you must ask me 48 hours before the due date of the homework* and have an excellent reason. After you have used up these late days, further late assignments will be penalized by 20% per day they are late.

Problem 1 (Large deviations for Brownian motion)**.** In this problem, you will derive the asymptotics of the very small probability that Brownian motion lies very close to the x-axis for a long time. Let $B(t)$ be a Brownian motion. Show that there is a constant $c > 0$ such that, for all $\epsilon > 0$ sufficiently small,

$$\mathbb{P}\left[\sup_{t\in[0,1]} |B(t)| \le \epsilon\right] \le \exp\left(-\frac{c}{\epsilon^2}\right).$$

You do not need to prove it, but it turns out that this is the correct behavior, up to the precise value of c.

(**Hint:** Consider a grid of n points t_i of the interval $[0,1]$. Show that, if $|B(t)| \le \epsilon$ for all $t \in [0,1]$, then $|B(t_i) - B(t_{i-1})|$ cannot be too large for any i. Estimate the probability that this happens and optimize over n.)

Problem 2 (Brownian bridge)**.** Let $B(t)$ be a Brownian motion. Define $R(t) := B(t) - tB(1)$. Note that $R(0) = R(1) = 0$. Intuitively, think of $R(t)$ as a Brownian motion “pinned” to take value 0 at time 1.

1. Show that $R(t)$ is a Gaussian process (i.e., that all of its finite distributions are Gaussian random vectors). Compute its mean function $\mu(t) = \mathbb{E}[R(t)]$ and covariance kernel $K(s,t) = \mathrm{Cov}[R(s), R(t)]$.

2. Let $X_1, X_2, \ldots$ be i.i.d. real-valued random variables that have distribution function $F(t) := \mathbb{P}[X_i \leq t]$. You may assume that F is strictly increasing. Note that F is a function $F : \mathbb{R} \to [0,1]$. Let $F_n : \mathbb{R} \to [0,1]$ be the empirical distribution function of n i.i.d. samples:

$$F_n(t) := \frac{\#\{i \in \{1,\ldots,n\} : X_i \leq t\}}{n}.$$

This is a random function, and thus we may view it as a continuous-time stochastic process. Define the associated normalized process

$$\widehat{F}_n(t) := \sqrt{n} \cdot (F_n(t) - F(t)).$$

Define a final process $X(t) := R(F(t))$, a "time change" of $R(t)$. Show that $(\widehat{F}_n(t))_{t \in \mathbb{R}}$ converges in finite distributions to $(X(t))_{t \in \mathbb{R}}$.

(**Hint:** Show that a vector of the form $(\widehat{F}_n(t_1), \ldots, \widehat{F}_n(t_k))$ may be viewed as a normalized sum of n i.i.d. random vectors having some covariance matrix $\mathbf{\Sigma} \in \mathbb{R}^{k \times k}$. Use a multidimensional central limit theorem (you may look this up and use it without proof) on this sum. and compare $\mathbf{\Sigma}$ with your computation of $K(s,t)$ above.)

Problem 3 (Ornstein-Uhlenbeck process)**.** Let B be a Brownian motion. Define the process $U(t)$ over all $t \in \mathbb{R}$ by

$$U(t) := e^{-t} B(e^{2t}).$$

1. Show that $U(t)$ is a Gaussian process. Compute its mean function $\mu(t) = \mathbb{E}[U(t)]$ and covariance kernel $K(s,t) = \mathrm{Cov}[U(s), U(t)]$. Conclude that $(U(t))_{t \in \mathbb{R}}$ has the same law (i.e., the same finite distributions) as $(U(c+t))_{t \in \mathbb{R}}$ for any $c \in \mathbb{R}$.

2. For any $t \geq s$, compute $\mathbb{E}[U(t) \mid U(s)]$ as a function (almost surely) of $U(s)$. Conclude that $U(t)$ is not a continuous-time martingale, in the sense that the value of the above is not $U(s)$.

   (**Hint:** Reduce this to a matter of conditioning one Gaussian variable on another. Consult Problem 5 of Homework 2.)

3. Derive a symmetry of the law of $U(t)$ from the time inversion symmetry of Brownian motion. Check that your conclusion is compatible with the calculation in Part 1.

4. Consider the random variable $X = \int_0^t U(s)\,ds$ for some $t \geq 0$. (This should not be mysterious—it is just an integral of a continuous function, where the function happens to be random.) Compute the mean and variance of X as functions of t.

Problem 4 (A first stochastic integral)**.** Let $B(t)$ be a Brownian motion.

1. Show that, almost surely,

$$\lim_{n \to \infty} \sum_{j=1}^{2^n} \left| B\left(\frac{j}{2^n}\right) - B\left(\frac{j-1}{2^n}\right) \right| = \infty.$$

2. Show that, almost surely,

$$\lim_{n\to\infty} \sum_{j=1}^{2^n} \left| B\left(\frac{j}{2^n}\right) - B\left(\frac{j-1}{2^n}\right) \right|^2 = 1.$$

3. Let $F : [0,1] \to \mathbb{R}$ be a smooth function. For each $n \geq 0$, define

$$I_n := \sum_{j=1}^{2^n} 2^n \cdot \left(F\left(\frac{j}{2^n}\right) - F\left(\frac{j-1}{2^n}\right)\right) \cdot \left(B\left(\frac{j}{2^n}\right) - B\left(\frac{j-1}{2^n}\right)\right).$$

Show that the I_n converge in L^2 to some (random) limit. Intuitively, you should view this limit as $\int_0^1 F' \, dB$, a stochastic Riemann-Stieltjes integral.

(**Hint:** Show that the I_n form a martingale and use a suitable convergence theorem.)