Lecture 20: Replica Symmetry Breaking II

1 Introduction

This lecture covers the the second half of the calculations of Giorgio Parisi's famous formula derived by use of replica symmetry breaking.

Proposition 1.1 (MGF of a Gaussian)**.**

$$\mathbb{E}_{g\sim\mathcal{N}(0,1)} \exp(ga) = \exp(a^2/2) \tag{1}$$

2 The RSB ansatz: Frobenius norm of Q

Recall that we left off with, using the replica trick,

$$\mathbb{E}Z(\beta)^n \approx \exp\left(N \sup_Q \inf_\Lambda \left[n\log 2 + \beta^2 \|Q\|_F^2 + \log \mathbb{E}_{y\sim Unif(\{1,-1\}^n)} \exp(y^T\Lambda y - \langle \Lambda, Q\rangle)\right]\right) \tag{2}$$

By taking matrix derivatives, one can solve for the minimizing Λ in the expression as $2\beta^2 Q$.

$$\mathbb{E}Z(\beta)^n \approx \exp\left(N \sup_Q \left[n\log 2 + \beta^2 \|Q\|_F^2 + n + \log \mathbb{E}_y \exp(2\beta^2 y^T Q y)\right]\right) \tag{3}$$

The replica symmetry breaking ansatz has parameters $n \geq m_1 \geq m_2 > ...m_k \geq 1$. Each a_i is the value of the block between each pair of levels. The idea of Parisi is that for an integer number of replicas, the entries of the replica matrix Q can take only k positive values.

$$Q_{i,j} = a_i \text{ if } \left[\frac{i}{m_k}\right] \neq \left[\frac{j}{m_k}\right] \text{ and } \left[\frac{i}{m_{k+1}}\right] = \left[\frac{j}{m_{k+1}}\right] \tag{4}$$

We need Q^* to be positive semidefinite. Physically, this corresponds to stability of the free energy under infinitesimal perturbations. So, we have a condition on the a_i's to make this happen. It turns out that Q^* is PSD if and only if $a_0 \leq a_1 \leq \cdots \leq a_k \leq 1$.

This block structure of Q^* allows us to directly calculate the Frobenius norm of Q^*.

$$\|Q^*\|_F^2 = n[1 + \sum_{j=0}^{k} a_j^2(m_j - m_{j+1})] \tag{5}$$

if we define $m_0 = n$ and $m_{k+1} = 1$.

Recall that $f(\beta) = \lim_{N\to\infty} \frac{1}{n} \lim_{n\to 0} \frac{1}{n} \log \mathbb{E}Z(\beta)^n$. As $n \to \infty$, $\frac{1}{n}\|Q^*\|_F^2 = 1 + \sum_{j=0}^{k} a_j^2(m_j - m_{j+1}$. *Weird physics alert:* Also as $n \to \infty$, the chain of inequalities $n \geq m_1 \geq \ldots ... \geq 1$

reverses to $m_0 \leq m_1 \leq \cdots \leq m_{k+1} = n$. See the Parisi paper. It has something to do with the symmetric group of 0 elements.

The sum in (5) can be interpreted as a Riemann sum as the $m_i's$ fill the interval; as $k \to \infty$. This sum turns into $1 - \int_0^1 a(x)^2 dx$.

So we have

$$\frac{1}{n} \|Q\|_F^2 = 1 - \int_0^1 a(x)^2 dx \tag{6}$$

In the replica symmetric version, we ended up with an optimization over a. Now, for each x, we optimize over $a(x)$. So we write the $\log \mathbb{E}_y \exp(2\beta^2 y^T Q y)$ term in terms of $a(x)$. In the literature, this function a (often seen as q), is called the *Parisi order parameter*.

3 RSB Ansatz: Entropy Term

Now we deal with the entropy term in (2).

Proposition 3.1.

$$\mathbb{E}_{y \sim Unif(\{\pm 1\}^n} \exp\left(\frac{1}{2} y^T \Sigma y + \langle \mu, y \rangle\right) = \mathbb{E}_{g \sim \mathcal{N}(\mu, \Sigma)} \prod_i \cosh(g_i) \tag{7}$$

Proof. Use the spectral decomposition of Σ, which we take to be positive definite.

$$\Sigma = V \Lambda V^T = \sum_{i=1}^n \lambda_i v_i v_i^T \tag{8}$$

Then the LHS becomes

$$\mathbb{E}_{y \sim Unif(\{\pm 1\}^n} \exp\left(\frac{1}{2} y^T \Sigma y + \langle \mu, y \rangle\right) = \mathbb{E}_y \exp\left(\frac{1}{2} \sum \lambda_i \langle v_i, y \rangle^2 + \langle \mu, y \rangle\right) \tag{9}$$

Now we use the proposition, the MGF of a Gaussian, because we can write the term in the exponential as $\sum_j y_j (\mu_j + \sum_i v_i \sqrt{\lambda_i} y_i)$. So we can write this as

$$\mathbb{E}_{g_i \sim N(0,1)} \exp(\sum_j y_j (\mu_j + \sum_i v_i \sqrt{\lambda_i} g_i)) = \mathbb{E}_{g \sim N(\mu, \Sigma)} \cosh(\mu_j + (V \sqrt{D} g)_j) \tag{10}$$

□

This yields the following recursion

$$\mathbb{E}_y \exp(\frac{1}{2} y^T Q^{(k-1)} y) = \mathbb{E}_{g \sim N(0,1)} \prod_i \cosh(g_i) \tag{11}$$

Let $\mathbb{1}^{(j)}$ be the matrix of j^2 square blocks that is one on the diagonal.

$Q^{(k)} = a_0 \mathbb{1}^{(1)} + (a_1 - a_0) \mathbb{1}^{(3)} + ...$

Starting from the first step RSB, we have, letting $g = h * 1 + g_1$,

$$\mathbb{E}_y \,\mathsf{exp}(2\beta^2 y^T Q y) = \mathbb{E}_{h\sim N(0,a_0)}\mathbb{E}_{g\sim N(0,\mathbb{1}^{(3)}}\prod_{i=1}^{n}\mathsf{cosh}(h+g_i) = \mathbb{E}_{h\sim N(0,a_0)}[\mathbb{E}_y \,\mathsf{exp}(\frac{1}{2}y^T Q^{(0)} y + h\mathbb{1}^T y)]^{(n/m_1)} \tag{12}$$

This procedure can be iterated. It is closely related to the heat equation, which comes up in Parisi's formula.