Assignment 2

CPSC 664 (Spring 2023)

Modern Probability for Theoretical Computer Science

Assigned: April 11, 2023 Due: May 4, 2023

Solve any two out of the three problems. If you solve them all, I will grade the first two. Each problem will be worth an equal amount towards your grade.

Problem 1 (Free probability)**.** Define the 2×2 matrix

$$\boldsymbol{A} := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{1}$$

Let $t \sim \mathsf{Unif}([0, \pi])$ and define the random rotation matrix

$$\boldsymbol{U} := \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}. \tag{2}$$

1. Show that $\boldsymbol{A}$ and $\boldsymbol{U}\boldsymbol{A}\boldsymbol{U}^\top$ are (exactly) free.
2. To what measure must the empirical spectral distribution of $\boldsymbol{A}+\boldsymbol{U}\boldsymbol{A}\boldsymbol{U}^\top$ then converge in moments? Why?
3. The empirical spectral distribution of $\boldsymbol{A} + \boldsymbol{U}\boldsymbol{A}\boldsymbol{U}^\top$ always consists of at most two atoms. Why? Therefore, it will never look like the measure you described in Part 2. Why is this not a contradiction?

Problem 2 (Sherrington-Kirkpatrick free energy)**.** Recall the definitions of the basic thermodynamic quantities associated to the Sherrington-Kirkpatrick model: $\boldsymbol{W} \sim \mathsf{GOE}(N)$ is normalized such that $\mathbb{E}\|\boldsymbol{W}\| \to 2$ as $N \to \infty$, and given $\boldsymbol{W}$ we set

$$Z(\beta) := \sum_{\boldsymbol{x} \in \{\pm 1\}^N} \exp(\beta \boldsymbol{x}^\top \boldsymbol{W} \boldsymbol{x}), \tag{3}$$

$$F(\beta) := \log Z(\beta), \tag{4}$$

$$f(\beta) := \lim_{N \to \infty} \frac{1}{N} \mathbb{E}_{\boldsymbol{W}} F(\beta). \tag{5}$$

1. Use Jensen's inequality and the moment generating function of $\boldsymbol{W}$ to prove a bound $f(\beta) \leq q(\beta)$ for $q(\beta)$ a quadratic function of β.

2. Consider the function $\widetilde{f}(\beta) := f(\beta)/\beta$. Rewrite your bound above as a bound on $\widetilde{f}(\beta)$. Show that $\widetilde{f}(\beta)$ is non-increasing in β. Use this to find a constant $C > 0$ such that, for all $\beta > C$, we must have $f(\beta) \neq q(\beta)$. (You will not prove this, but in fact this equality does hold for sufficiently small β, i.e., at sufficiently high temperature.)

3. Use the above to prove an upper bound on the Parisi number $2P_* = \lim_{\beta\to\infty} \widetilde{f}(\beta)$ that is strictly smaller than 2. (Recall that the upper bound of 2 follows from $\mathbb{E}\|\boldsymbol{W}\| \to 2$.)

Problem 3 (Min-max comparison inequality)**.** Let $\boldsymbol{\Sigma}^{(a)} \in \mathbb{R}^{mn\times mn}$ for $a \in \{0,1\}$ and $[mn]$ identified with $[m]\times[n]$ satisfy the following properties:

$$\Sigma^{(0)}_{ij,ij} = \Sigma^{(1)}_{ij,ij} \text{ for all } i \in [m], j \in [n], \tag{6}$$

$$\Sigma^{(0)}_{ij,ik} \geq \Sigma^{(1)}_{ij,ik} \text{ for all } i \in [m], j,k \in [n], \tag{7}$$

$$\Sigma^{(0)}_{ij,\ell k} \leq \Sigma^{(1)}_{ij,\ell k} \text{ for all } i,\ell \in [m], j,k \in [n] \text{ with } i \neq \ell. \tag{8}$$

Following the proof of the (weak) Sudakov-Fernique inequality from class, prove that, in this case,

$$\mathop{\mathbb{E}}_{\boldsymbol{g}\sim\mathcal{N}(\boldsymbol{0},\boldsymbol{\Sigma}^{(0)})}\left[\min_{1\leq i\leq m}\max_{1\leq j\leq n} g_{ij}\right] \leq \mathop{\mathbb{E}}_{\boldsymbol{g}\sim\mathcal{N}(\boldsymbol{0},\boldsymbol{\Sigma}^{(1)})}\left[\min_{1\leq i\leq m}\max_{1\leq j\leq n} g_{ij}\right]. \tag{9}$$

Note that this generalizes the Sudakov-Fernique inequality, which is the case $m = 1$.