Assignment 3

CPSC 663: Sum-of-Squares Optimization (Spring 2022)

Assigned: April 11, 2022   Due: May 6, 2022

Read the instructions for your project presentation, solve Problem 1, and solve any two out of the three remaining problems. If you solve them all, I will grade Problems 2 and 3. Each problem will be worth an equal amount towards your grade.

Project Presentation You will present your final project on Wednesday, April 27, 2022, the date of our last class meeting. You will present on the whiteboard and should plan to speak for about 25 minutes, with 5 minutes for questions or discussion at the end. Please rehearse your presentation in full to make sure you are within the time limits.

Problem 1 (Project report)**.** Write 1-2 pages (more is fine, but be reasonable) about the topic of your final project. First, outline the material you cover in your presentation. Then, reflect on what you learned from reading your paper or working on your problem. What are the main technical ideas of what you read or did? Where else could they be useful? Did you find anything surprising? Are there any open problems remaining in the particular topic you worked on that you find interesting?

Problem 2 (Second moment method)**.** The goal this problem is to show that, if $\epsilon > 0$ and $G \sim \mathcal{G}(n, \frac{1}{2})$, then with high probability $\omega(G) \geq (2 - \epsilon) \log_2 n$.

1. Let X be a random variable taking values in the non-negative real numbers and having finite second moment. Use the Cauchy-Schwarz inequality to show that

$$\mathbb{P}[X > 0] \geq \frac{(\mathbb{E}[X])^2}{\mathbb{E}[X^2]}. \tag{1}$$

2. Show that, for any $k \leq n/2$ and $s \leq k$, we have

$$\frac{\binom{k}{s}\binom{n-k}{k-s}}{\binom{n}{k}} \leq \binom{k}{s}\left(\frac{k}{n}\right)^s \leq \left(\frac{k^2}{n}\right)^s. \tag{2}$$

   The terms $\binom{k}{s}$ on either side may be cancelled, but written this way the statement has a nice probabilistic interpretation: the left-hand side is the probability that a uniformly random subset of $[n]$ of size exactly k contains exactly s of the first k elements, while the middle expression is the same but for a random subset where each element in $[n]$ is chosen independently with probability $\frac{k}{n}$ (approximately, when $k \ll n$).

3. Let $k = (2 - \epsilon) \log_2 n$, and let X be the number of k-cliques in G. Use the above results to show that $X > 0$ with high probability. In particular, use Part 1, and then expand $\mathbb{E}[X^2]$, group terms according to the size of the intersection of two "potential cliques" in G, and use Part 2.

Problem 3 (Random matrix norm)**.** Let $G \in \mathbb{R}^{n \times n}_{\mathsf{sym}}$ have $G_{ij} = G_{ji} \sim \mathsf{Unif}(\{\pm 1\})$ independently for $i \neq j$, and $G_{ii} = 0$. The goal of this problem is to show that $\mathbb{E}\|G\| = O(\sqrt{n} \cdot \log n)$ (a suboptimal result, but only up to the $\log n$ factor).

1. Show that $\mathbb{E}\|G\| \leq (\mathbb{E}\,\mathsf{Tr}(G^{2k}))^{1/2k} =: T_{2k}^{1/2k}$ for any $k \geq 1$.
2. Describe a subset $S_{2k} \subseteq [n]^{2k}$ such that $T_{2k} = |S_{2k}|$. Give an alternative description in terms of walks in the complete graph K_n with vertices labelled by $[n]$.
3. Show that $|S_{2k}| \leq k^{2k} n^{k+1}$ (it may help to convince yourself first that the biggest contribution comes from those elements of S with as many different entries as possible, i.e., corresponding to walks touching as many different vertices as possible).
4. Choose k appropriately to obtain the conclusion.

Problem 4 (Spider shapes)**.** Consider the following special type of graphical matrix for a shape U: let A, B, C be disjoint, with $|A| = |B| = a$ and $|C| = 1$, and let there be $2a$ edges in U, connecting every vertex in either A or B to the one vertex in C. Some papers have called this kind of shape a *spider*; you can see the reason if you draw a picture of the kind we have been drawing in class.

1. Show that $\widetilde{\boldsymbol{M}}^{(U)} \succeq \boldsymbol{0}$. Describe a general class of shapes U for which a similar argument shows positive semidefiniteness (you do not have to be too precise).
2. Give a matrix $\boldsymbol{H} \in \mathbb{R}^{n \times n}_{\mathsf{sym}}$ whose spectrum, ignoring zero eigenvalues, is the same as that of $\widetilde{\boldsymbol{M}}^{(U)} \in \mathbb{R}^{[n]^a \times [n]^a}_{\mathsf{sym}}$ (evaluated with the ± 1 adjacency matrix $\boldsymbol{G}$ of a graph G). You should be able to describe the entries of $\boldsymbol{H}$ explicitly in terms of those of $\boldsymbol{G}$. Note that this implies $\mathsf{rank}(\widetilde{\boldsymbol{M}}^{(U)}) \leq n$. Show also that, for general shapes U, $\mathsf{rank}(\widetilde{\boldsymbol{M}}^{(U)}) \leq n^{\mathsf{sep}(U)}$ (again, generalize your reasoning for spiders, and you do not have to be too precise in this part).
3. Show that $\|\widetilde{\boldsymbol{M}}^{(U)}\| = \|\boldsymbol{H}\| \leq \|\boldsymbol{G}\|^{2a}$. Combined with the result of the previous problem, this confirms for the special case of spiders the general graphical matrix norm bound.