Connections between structured tight frames and sum-of-squares optimization

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ABSTRACT

This note describes a new technique for generating tight frames that have a high degree of symmetry and entrywise Gramian structure. The technique is based on a "lifting" construction through which, from non-maximal real equiangular tight frames of N vectors in r-dimensional space, we produce real unit-norm tight frames of $\binom{N}{2}$ vectors in $(\frac{r(r+1)}{2} - N)$-dimensional space. While these lifted frames are usually not equiangular, they do tend to inherit the property of having few distinct mutual angles, suggesting that the lifted frames are themselves interesting geometric objects. The Gram matrices of these frames are multiples of projection operators that also play an important role in degree 4 sum-of-squares relaxations of combinatorial problems such as MaxCut. We describe this motivation for our construction, and suggest several directions in which related ideas might be extended in future work.

Keywords: Finite frames, equiangular tight frames, sum-of-squares, combinatorial optimization

1. INTRODUCTION

Sets of unit vectors in $\mathbb{R}^r$ whose mutual angles take on only a small number of distinct values have been widely studied, and they are known to be intimately related to combinatorial structures such as tight frames, spherical designs, strongly regular graphs, and other combinatorial designs.[1–5] It is therefore of interest to produce new constructions and examples of such objects. Often, such constructions arise from new ideas pertaining to the underlying combinatorics of these geometric objects. In this note, we present a different idea for producing new highly structured sets of unit vectors, which proceeds by "lifting" existing such sets to include more and higher-dimensional vectors.

This lifting procedure involves an interesting mix of ideas, stemming from the recent demonstration in a work[6] of the authors' that *equiangular tight frames*, one type of highly structured vector collection, may be used to produce feasible points for a certain semidefinite programming relaxation, a *sum-of-squares relaxation*, of an optimization problem. This amounts to producing a large positive semidefinite matrix with intricate entrywise structure involving symmetries and repetitions of the Gram matrix entries of the initial equiangular tight frame. By adjusting this matrix slightly, we are able to produce a new matrix, now the Gram matrix of a *unit-norm tight frame*, which still retains much of the entrywise structure of the previous matrix related to sum-of-squares programming. In particular, it appears that the resulting unit-norm tight frames typically have few mutual angles among their vectors, and that the graphs indicating which vectors share the same mutual angle sometimes enjoy special structure. We propose this construction as an interesting new source of examples of structured tight frames, and suggest several directions in which it might be further explored.

The remainder of the note is organized as follows. In Section 2, we introduce the necessary background information: definitions of unit-norm and equiangular tight frames, some objects from combinatorial and sum-of-squares optimization, the key idea from convex geometry describing the row space of the unit-norm tight frame we construct, and a result from a previous work of the authors' that connects these topics together. In Section 3,

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we describe explicitly the unit-norm tight frame construction we propose, and give a formula for its Gram matrix entries in terms of the Gram matrix entries of the underlying equiangular tight frame. In Section 4, we work through the simple example of *simplex ETFs*, which turns out to produce via our construction a unit-norm tight frame associated with the *Johnson graph*. In Section 5, we suggest a further empirical regularity pertaining to the entrywise Gramian structure of the unit-norm tight frames we construct. Finally, in Section 6, we give some open problems that may be of interest in pursuing this topic further.

2. PRELIMINARIES

2.1 Structured Unit-Norm Frames

We first review certain structured types of *frames* in finite dimension. These are the objects that we will later describe a novel method for producing. In finite dimension, a *unit-norm frame* is merely a spanning set of unit vectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_N \in \mathbb{R}^r$. Throughout we will also write $\boldsymbol{V} \in \mathbb{R}^{r \times N}$ for the matrix having the $\boldsymbol{v}_i$ as its columns. The following are important specializations of this notion that describe various extra structure.

DEFINITION 2.1. *A unit-norm frame $\boldsymbol{v}_1, \dots, \boldsymbol{v}_N$ forms a* unit-norm tight frame (UNTF) *if either of the following equivalent conditions holds:*

1. $\sum_{i=1}^N \boldsymbol{v}_i \boldsymbol{v}_i^\top = \frac{N}{r} \boldsymbol{I}_r$.
2. *The Gram matrix $\boldsymbol{V}^\top \boldsymbol{V}$ equals an orthogonal projection matrix multiplied by $\frac{N}{r}$.*

DEFINITION 2.2. *A unit-norm tight frame $\boldsymbol{v}_1, \dots, \boldsymbol{v}_N$ forms an* equiangular tight frame (ETF) *if there exists a real constant $\mu > 0$ such that $|\langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle| = \mu$ whenever $i \neq j$.*

The following are two essential facts about ETFs. First, ETFs are precisely those unit-norm frames that minimize the *worst-case coherence*, the largest absolute inner product between two frame vectors. As a consequence, for fixed N and r, the constant μ has a single possible value.

PROPOSITION 2.3 (WELCH BOUND[7]). *If $\boldsymbol{v}_1, \dots, \boldsymbol{v}_N \in \mathbb{R}^r$ with $\|\boldsymbol{v}_i\|_2 = 1$, then*

$$\max_{\substack{1 \le i,j \le N \\ i \neq j}} |\langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle| \geq \sqrt{\frac{N - r}{r(N-1)}}, \tag{1}$$

with equality if and only if $\boldsymbol{v}_1, \dots, \boldsymbol{v}_N$ form an ETF.

Second, there is a bound on the size N of an ETF in terms of the dimension r of the frame vectors. Indeed, the bound applies to arbitrary equiangular unit vectors, and does not make use of the tight frame propery.

PROPOSITION 2.4 (GERZON BOUND[8]). *If $\boldsymbol{v}_1, \dots, \boldsymbol{v}_N \in \mathbb{R}^r$ with $\|\boldsymbol{v}_i\|_2 = 1$ for all $i \in [N]$ and $|\langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle| = \mu < 1$ for all $i, j \in [N]$ with $i \neq j$, then $N \leq \frac{r(r+1)}{2}$.*

ETFs are quite rare and highly structured; most known constructions are based on underlying combinatorial objects.[9–11] Cases where equality is achieved in the Gerzon bound are even rarer, with only finitely many such ETFs known to exist and infinitely many values of r for which the question of existence is open. The "living document" survey of Fickus and Mixon[11] gives a concise summary of both issues.

2.2 The Elliptope and its Sum-of-Squares Generalizations

Next, we describe several convex sets of matrices, which contain various subsets of the Gram matrices of unit norm frames (and their structured variants). These sets are important to combinatorial optimization, and have been thoroughly studied in that literature.[12–16]

DEFINITION 2.5. *The* cut polytope *in dimension N is the set*

$$\mathscr{C}^N := \mathsf{conv}\left(\left\{\boldsymbol{x}\boldsymbol{x}^\top : \boldsymbol{x} \in \{\pm 1\}^N\right\}\right) \subset \mathbb{R}^{N \times N}_{\mathsf{sym}}. \tag{2}$$

Definition 2.6. *The* elliptope *in dimension N is the set*

$$\mathscr{E}_2^N := \{\boldsymbol{X} \in \mathbb{R}^{N \times N}_{\mathsf{sym}} : \boldsymbol{X} \succeq \boldsymbol{0}, \mathsf{diag}(\boldsymbol{X}) = \boldsymbol{1}_N\} \subset \mathbb{R}^{N \times N}_{\mathsf{sym}}. \tag{3}$$

(Note that the elliptope is by definition the set of *all* Gram matrices of unit norm frames of N vectors in any dimension.)

The relationship between the cut polytope and the elliptope is the inclusion $\mathscr{C}^N \subseteq \mathscr{E}_2^N$. This simple fact has significant algorithmic implications for combinatorial optimization: if we want to solve the non-convex and in general NP-hard[17] optimization problem

$$\mathsf{M}(\boldsymbol{A}) := \max_{\boldsymbol{x} \in \{\pm 1\}^N} \boldsymbol{x}^\top \boldsymbol{A} \boldsymbol{x}, \tag{4}$$

then we may obtain a tractable approximation by forming the *relaxation* to the elliptope,

$$\mathsf{M}(\boldsymbol{A}) = \max_{\boldsymbol{X} \in \mathscr{C}^N} \langle \boldsymbol{A}, \boldsymbol{X} \rangle \le \max_{\boldsymbol{X} \in \mathscr{E}_2^N} \langle \boldsymbol{A}, \boldsymbol{X} \rangle, \tag{5}$$

the right-hand side of which is a semidefinite program that may be solved in time polynomial in N. The problem (4) includes important tasks such as finding the maximum cut in a graph, and the approach outlined above (along with certain *rounding* schemes for recovering a feasible point $\boldsymbol{x} \in \{\pm 1\}^N$ from a relaxed point $\boldsymbol{X} \in \mathscr{E}_2^N$) has had great success both in theory and in practice for solving such problems.[18–21]

The *sum-of-squares method* is an approach to algorithm design that provides a vast generalization of the above technique. We present a simple derivation of its application to the problem $\mathsf{M}(\boldsymbol{A})$ here, and direct the interested reader to numerous recent surveys[22–24] for more information on the general framework, as well as the comprehensive book of Blekherman, Parrilo, and Thomas[25] for a thorough reference.

The elliptope may be thought of as formed in the following way: we generate constraints on the matrix $\boldsymbol{X}$ by retaining positive semidefiniteness and linear constraints the matrix $\boldsymbol{x}\boldsymbol{x}^\top$ must satisfy when $\boldsymbol{x} \in \{\pm 1\}^N$, and discarding all other information (including in particular the fact that this matrix has rank one). Following this formulation, one may perform the same process on the larger matrix $(\boldsymbol{x} \otimes \boldsymbol{x})(\boldsymbol{x} \otimes \boldsymbol{x})^\top \in \mathbb{R}^{N^2 \times N^2}_{\mathsf{sym}}$. This matrix contains the degree 4 monomials $x_i x_j x_k x_\ell$, and therefore must satisfy a number of additional constraints. It also contains $\boldsymbol{x}\boldsymbol{x}^\top$ (repeatedly) as a principal minor, for instance in those entries where $i = j = 1$. This defines the following relaxation of $\mathscr{C}^N$, the *degree 4 sum-of-squares relaxation*.

Definition 2.7. *Let $\boldsymbol{Y} \in \mathbb{R}^{N^2 \times N^2}_{\mathsf{sym}}$, with the row and column indices of $\boldsymbol{Y}$ identified with pairs $(ij) \in [N]^2$, ordered lexicographically. Then, $\boldsymbol{Y}$ is a* degree 4 pseudomoment matrix* *if the following conditions hold:*

1. $\boldsymbol{Y} \succeq \boldsymbol{0}$.
2. $Y_{(ij)(kk)}$ *does not depend on the index k.*
3. $Y_{(ii)(ii)} = 1$ *for every $i \in [N]$.*
4. $Y_{(ij)(k\ell)}$ *is invariant under permutations of the indices i, j, k, ℓ.*

In this case, the matrix $\boldsymbol{Y}$ is said to extend *$\boldsymbol{X} \in \mathbb{R}^{N \times N}_{\mathsf{sym}}$ whose entries are given by $X_{ij} = Y_{(11)(ij)}$, and we write $\boldsymbol{X} \in \mathscr{E}_4^N$ if it admits an extension to a degree 4 pseudomoment matrix. The set $\mathscr{E}_4^N$ defined in this way is the* degree 4 generalized elliptope.

It follows that $\mathscr{C}^N \subseteq \mathscr{E}_4^N \subseteq \mathscr{E}_2^N$, and in fact for $N \geq 5$ both inclusions are strict; i.e., the degree 4 sum-of-squares relaxation is a strict improvement on the elliptope relaxation for some problems, but is also itself sometimes an imperfect approximation.[26,27] Relatively little specific information is known about the geometry of $\mathscr{E}_4^N$, which led the authors to study in a recent work[6] explicit examples of members of $\mathscr{E}_4^N$ and their degree 4 extensions' structure.

*The term "pseudomoment matrix" in this definition refers to the alternative point of view that $\boldsymbol{Y}$ contains fictitious degree 4 moments of a probability distribution over $\{\pm 1\}^N$; these *pseudomoments* are constrained to satisfy some but not all of the requirements of true moments. More details on this idea are given in the surveys of Lasserre and Laurent.[22,23]

2.3 Some Definitions from Convex Geometry

The constructions we will consider are related to certain ancillary features of the convex set $\mathscr{E}_2^N$. We recall the relevant general notions from convex geometry, which we will apply to $\mathscr{E}_2^N$. In what follows, let $K \subseteq \mathbb{R}^d$ be a closed convex set.

Definition 2.8. *The* dimension *of K is the dimension of the affine hull of K, denoted* $\dim(K)$.

Definition 2.9. *A convex subset $F \subseteq K$ is a* face *of K if whenever $\theta \boldsymbol{X} + (1-\theta)\boldsymbol{Y} \in F$ with $\theta \in (0,1)$ and $\boldsymbol{X}, \boldsymbol{Y} \in K$, then $\boldsymbol{X}, \boldsymbol{Y} \in F$.*

Definition 2.10. *$\boldsymbol{X} \in K$ is an* extreme point *of K if $\{\boldsymbol{X}\}$ is a face of K (of dimension zero).*

Definition 2.11. *The intersection of all faces of K containing $\boldsymbol{X} \in K$ is the unique* smallest face of K containing $\boldsymbol{X}$*, denoted* $\mathsf{face}_K(\boldsymbol{X})$.

Definition 2.12. *The* perturbation of $\boldsymbol{X}$ in K *is the subspace*

$$\mathsf{pert}_K(\boldsymbol{X}) := \left\{ \boldsymbol{A} \in \mathbb{R}^d : \boldsymbol{X} \pm t\boldsymbol{A} \in K \text{ for all } t > 0 \text{ sufficiently small} \right\}. \tag{6}$$

The perturbation will come up naturally in our results, so we present the following useful fact giving its connection to the more intuitive objects from facial geometry.

Proposition 2.13. *Let $\boldsymbol{X} \in K$. Then,*

$$\mathsf{face}_K(\boldsymbol{X}) = K \cap (\boldsymbol{X} + \mathsf{pert}_K(\boldsymbol{X})). \tag{7}$$

In particular, the affine hull of $\mathsf{face}_K(\boldsymbol{X})$ is $\boldsymbol{X} + \mathsf{pert}_K(\boldsymbol{X})$, and therefore

$$\dim(\mathsf{face}_K(\boldsymbol{X})) = \dim(\mathsf{pert}_K(\boldsymbol{X})) \tag{8}$$

(in which there is a harmless reuse of notation between the dimension of a convex set and the dimension of a subspace).

A proof of this result, which is probably folklore, was included in the previous work[6] of the authors'.

2.4 Degree 4 Extensions of Equiangular Tight Frames

As mentioned before, it is interesting to find concrete instances where degree 4 extensions of matrices of $\mathscr{E}_2^N$ may be written down explicitly. Surprisingly, ETFs turn out to be an excellent source of such examples of degree 4 extensions. The following theorem shows that ETFs in fact always admit degree 4 extensions, except in the maximal case $N = \frac{r(r+1)}{2}$ (we refer to this case as maximal per the Gerzon bound, our Proposition 2.4; recall from the discussion following the Proposition that this case is extremely rare). These extensions have an interesting structure both spectrally and entrywise.

Theorem 2.14 (Theorem 2.19 of "A Gramian Description..."[6]). *Let $\boldsymbol{v}_1, \dots, \boldsymbol{v}_N \in \mathbb{R}^r$ form an ETF, and let $\boldsymbol{X} \in \mathbb{R}^{N\times N}_{\mathsf{sym}}$ be the Gram matrix. Then, $\boldsymbol{X} \in \mathscr{E}_4^N$ if and only if $N < \frac{r(r+1)}{2}$. In this case, an extension $\boldsymbol{Y} \in \mathbb{R}^{N^2 \times N^2}$ is given by*

$$\boldsymbol{Y} = \mathsf{vec}(\boldsymbol{X})\mathsf{vec}(\boldsymbol{X})^\top + \frac{N^2(1-\frac{1}{r})}{\frac{r(r+1)}{2} - N} \boldsymbol{P}_{\mathsf{vec}(\mathsf{pert}_{\mathscr{E}_2^N}(\boldsymbol{X}))}, \tag{9}$$

with entries given by

$$Y_{(ij)(k\ell)} = \frac{\frac{r(r-1)}{2}}{\frac{r(r+1)}{2} - N}(X_{ij}X_{k\ell} + X_{ik}X_{j\ell} + X_{i\ell}X_{jk}) - \frac{r^2(1-\frac{1}{N})}{\frac{r(r+1)}{2} - N} \sum_{m=1}^{N} X_{im}X_{jm}X_{km}X_{\ell m}. \tag{10}$$

Note that, implicitly, Theorem 2.14 also contains a formula for the entries of the matrix $\boldsymbol{P}_{\mathsf{vec}(\mathsf{pert}_{\mathscr{E}_2^N}(\boldsymbol{X}))}$. It is this aspect of the result that we will pursue in the following section in order to "lift" an ETF to a new UNTF using this construction.

3. THE DEGREE 4 LIFTING

Our construction begins by noting that $\boldsymbol{Y}$ is a positive semidefinite matrix whose diagonal identically equals one, and which is *almost* a projection matrix, with the caveat of the presence of the rank-one component $\mathsf{vec}(\boldsymbol{X})\mathsf{vec}(\boldsymbol{X})^\top$. This observation suggests that we might adjust $\boldsymbol{Y}$ to form the Gram matrix of a UNTF.

We begin by subtracting the rank one component, forming $\boldsymbol{Y} - \mathsf{vec}(\boldsymbol{X})\mathsf{vec}(\boldsymbol{X})^\top$.[†] We compute some of the entries of this matrix:

$$(\boldsymbol{Y} - \mathsf{vec}(\boldsymbol{X})\mathsf{vec}(\boldsymbol{X})^\top)_{(ii)(jj)} = Y_{(ii)(jj)} - X_{ii}X_{jj} = 0 \text{ for any } i, j \in [N], \tag{11}$$

$$(\boldsymbol{Y} - \mathsf{vec}(\boldsymbol{X})\mathsf{vec}(\boldsymbol{X})^\top)_{(ij)(ij)} = Y_{(ij)(ij)} - X_{ij}^2 = 1 - \mu^2 \text{ for } i \neq j. \tag{12}$$

Thus, to remove the large identically zero submatrix, we discard from this matrix the rows and columns indexed by pairs (ii). Among the remaining rows and columns, those indexed by (ij) and (ji) are equal, so we further restrict our attention to pairs (ij) with $i < j$, which we henceforth denote $\{ij\}$ to emphasize that we think of them as unordered pairs. This matrix is still a projection matrix; indeed, viewing the action of $\boldsymbol{P}_{\mathsf{vec}(\mathsf{pert}_{\mathscr{E}_2^N}(\boldsymbol{X}))}$ as projecting to $\mathsf{pert}_{\mathscr{E}_2^N}(\boldsymbol{X})$ in the Hilbert space $\mathbb{R}^{N\times N}_{\mathsf{sym}}$ of $N \times N$ symmetric matrices, we have merely restricted to the action on symmetric matrices whose diagonal is equal to zero, which is a linear subspace of $\mathbb{R}^{N\times N}_{\mathsf{sym}}$ containing $\mathsf{pert}_{\mathscr{E}_2^N}(\boldsymbol{X})$. Renormalizing by $1 - \mu^2 = \frac{N(r-1)}{r(N-1)} = \frac{1-r^{-1}}{1-N^{-1}}$, the following characterization of the matrix we have constructed then follows.

Corollary 3.1. *Let $\boldsymbol{X}$ be the Gram matrix of an ETF of N vectors in $\mathbb{R}^r$, with $N < \frac{r(r+1)}{2}$. Define the matrix $\boldsymbol{M} = \boldsymbol{M}(\boldsymbol{X}) \in \mathbb{R}^{\binom{N}{2}\times\binom{N}{2}}$ to have entries*

$$M_{\{ij\}\{k\ell\}} = \frac{1-\frac{1}{N}}{(1-\frac{1}{r})(\frac{r(r+1)}{2} - N)}\Bigg[(N-r)X_{ij}X_{k\ell} + \frac{r(r-1)}{2}(X_{ik}X_{j\ell} + X_{i\ell}X_{jk}) - r^2\left(1 - \frac{1}{N}\right)\sum_{m=1}^{N} X_{im}X_{jm}X_{km}X_{\ell m}\Bigg] \tag{13}$$

Then, $\boldsymbol{M}$ is the Gram matrix of a UNTF. The rank of $\boldsymbol{M}$ (equivalently, the minimum dimension that may be taken for the underlying UNTF vectors) is $\frac{r(r+1)}{2} - N$.

4. EXAMPLE: SIMPLEX ETFS AND JOHNSON GRAPHS

In this section, we describe a simple example where the structure of the UNTF described by Corollary 3.1 can be described explicitly and in closed form. We will consider the application of Corollary 3.1 to the *simplex ETF*, an ETF of N vectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_N \in \mathbb{R}^r$ with $r = N - 1$. The frame vectors point to the vertices of an equilateral simplex (whose side length is determined by the unit length of the frame vectors), and it may then be computed that the inner product of any two frame vectors is $-\frac{1}{N-1}$, whereby $\mu = \frac{1}{N-1}$. Thus the Gram matrix of the simplex ETF in dimension $N-1$ is given by

$$\boldsymbol{X}^{(N)} = \left(1 + \frac{1}{N-1}\right)\boldsymbol{I}_N - \frac{1}{N-1}\boldsymbol{1}_N\boldsymbol{1}_N^\top \in \mathbb{R}^{N\times N}_{\mathsf{sym}}. \tag{14}$$

When $N = 3$, then $r = 2$ and $N = \frac{r(r+1)}{2}$, so the simplex ETF is maximal. Thus let us restrict our attention to the case $N \geq 4$, in which case Corollary 3.1 applies to the simplex ETF.

Let $\boldsymbol{M}^{(N)} \in \mathbb{R}^{\binom{N}{2}\times\binom{N}{2}}$ be the UNTF Gram matrix produced by Corollary 3.1. Note that since, in the case of the simplex ETF, X_{ij} only depends on whether $i = j$ or $i \neq j$, and the entries of $\boldsymbol{M}$ are polynomials in the X_{ij}, the value of entry $M^{(N)}_{\{ij\}\{k\ell\}}$ will depend only on the equalities among the indices i, j, k, ℓ. Since we have

[†] Per the "pseudomoment" interpretation of $\boldsymbol{Y}$ and $\boldsymbol{X}$ mentioned previously, the matrix $\boldsymbol{Y} - \mathsf{vec}(\boldsymbol{X})\mathsf{vec}(\boldsymbol{X})^\top$ may be viewed as a "pseudocovariance matrix" of degree 2 monomials.

$i \neq j$ and $k \neq \ell$, in fact the value of $M^{(N)}_{\{ij\}\{k\ell\}}$ will depend only on the size of the intersection of the index sets, $|\{i,j\} \cap \{k,\ell\}|$. This property means that $\boldsymbol{M}^{(N)}$ belongs to a family of $\binom{N}{2} \times \binom{N}{2}$ matrices known as the *Johnson association scheme*, whose definition is simply that described above, that the entries depend only on the intersection sizes of the index sets.[28]

While the Johnson association scheme may be defined in the same way for $\binom{N}{k} \times \binom{N}{k}$ matrices for any $k \in [N]$, in the case $k = 2$ its structure admits an elegant interpretation in terms of a graph.

DEFINITION 4.1. *The* Johnson graph of order N *is the graph on* $\binom{N}{2}$ *vertices, indexed by subsets of* $[N]$ *having size 2, such that* $\{i,j\} \sim \{k,\ell\}$ *if and only if* $|\{i,j\} \cap \{k,\ell\}| = 1$.

Let us write $\boldsymbol{A}^{(N)} \in \mathbb{R}^{\binom{N}{2} \times \binom{N}{2}}$ for the adjacency matrix of the Johnson graph of order N, and $\bar{\boldsymbol{A}}^{(N)}$ for the adjacency matrix of its complement. Then, being an element of the Johnson association scheme for $k = 2$ is equivalent to being a linear combination of $\boldsymbol{I}_N$, $\boldsymbol{A}^{(N)}$, and $\bar{\boldsymbol{A}}^{(N)}$. These matrices are simultaneously diagonalizable over the eigenspaces of $\boldsymbol{A}^{(N)}$, and there are only three of these eigenspaces, a feature of the Johnson graph being a *strongly regular graph*.[28,30] The following then follows from a straightforward computation.

PROPOSITION 4.2. *The Johnson graph of order N has a unique negative eigenvalue, equal to -2, whose corresponding eigenspace has dimension $\frac{N(N-3)}{2}$. The degree 4 lifting $\boldsymbol{M}^{(N)}$ of the simplex ETF of N vectors is then a constant multiple of the projection matrix onto this eigenspace.*

$\boldsymbol{M}^{(N)}$ is therefore also equal to the Gram matrix of the two-distance tight frame corresponding to the Johnson graph under the correspondence of Barg et al.[5] Namely, $\boldsymbol{M}^{(N)}$ is uniquely determined by having the following properties: (1) it is the Gram matrix of a UNTF, (2) the positions of the entries of $\boldsymbol{M}^{(N)}$ equal to some $\alpha \in \mathbb{R}$ give precisely the adjacencies in the Johnson graph, and (3) the remaining off-diagonal entries take on a single different value $\beta \in \mathbb{R} \setminus \{\alpha\}$.

In summary, in this section we have observed several manifestations of the following informal statement:

> *The degree 4 lifting applied to the simplex ETF produces a UNTF whose spectral and entrywise structure is connected to the Johnson graph.*

The simplex ETF, however, is among the simplest of ETFs. In the following section, we make an empirical observation that suggests that such connections with structured graphs are in fact a more general property of the degree 4 lifting.

5. EMPIRICAL REGULARITY OF ENTRY GRAPHS

The discussion in the previous section motivates the following definition.

DEFINITION 5.1. *Let $\boldsymbol{X} \in \mathbb{R}^{N \times N}_{\mathsf{sym}}$ be the Gram matrix of an ETF of N vectors in $\mathbb{R}^r$ with $N < \frac{r(r+1)}{2}$, and let $\boldsymbol{M}$ be the Gram matrix of the UNTF constructed by the degree 4 lifting from $\boldsymbol{X}$. For $\alpha \geq 0$, the* entry graph of α in $\boldsymbol{M}$ *is the graph on $\binom{N}{2}$ vertices where $i \sim j$ if $|M_{ij}| = \alpha$.*

(It may appear more natural to consider this definition with an equality condition on M_{ij} instead of $|M_{ij}|$, but it appears that the structure we observe only occurs in the latter case, whereby this seems to be the "correct" definition.)

For instance, in Section 4, we observed that the entry graphs of the degree 4 lifting of the simplex ETF are the Johnson graph, its complement, and the graph without edges (when $\alpha = 1$). Examining entry graphs for other ETFs by hand, we make the following observation.

> *For the degree 4 liftings of ETFs belonging to certain families, all entry graphs are regular graphs.*

These families seem to include, for instance, Paley ETFs, as well as ETFs corresponding, under the correspondence (described by Fickus and Watson[29] and others) between ETFs and strongly regular graphs, to graphs built from orthogonal polar spaces.[30] On the other hand, numerical experiments on ETFs corresponding in the same way to other strongly regular graphs (as obtained from the `SageMath` software package) show that their degree 4 liftings do not satisfy this property, showing that the regularity of entry graphs is not a universal property of degree 4 liftings of ETFs.

6. OPEN PROBLEMS

To conclude, we suggest two directions in which the ideas presented here might be fruitfully pursued. First, the sum-of-squares "hierarchy" of algorithms suggests a way to extend the lifting discussed here to even higher-dimensional liftings and associated UNTFs.

QUESTION 6.1. *Define the* higher-degree generalized elliptopes *by repeating the procedure of Section 2.2, now requiring for $\boldsymbol{X} \in \mathscr{E}_{2k}^N$ that there exist an $N^k \times N^k$ matrix $\boldsymbol{Y}$ satisfying the linear and positivity constraints of $(\boldsymbol{x}^{\otimes k})(\boldsymbol{x}^{\otimes k})^\top$ (see the authors' work*[6] *for a more formal definition of these sets). Several questions extending those discussed here are then natural:*

1. *When does the Gram matrix $\boldsymbol{X}$ of an ETF belong to $\mathscr{E}_{2k}^N$?*
2. *When indeed $\boldsymbol{X} \in \mathscr{E}_{2k}^N$ and $\boldsymbol{Y} \in \mathbb{R}_{\mathsf{sym}}^{N^k \times N^k}$ is the "witness" of this membership, how does the entrywise and spectral structure $\boldsymbol{Y}$ relate to $\boldsymbol{X}$?*
3. *Do the orthogonal projection matrices to eigenspaces of $\boldsymbol{Y}$, after a suitable adjustment, form Gram matrices of UNTFs?*
4. *If so, what is the entrywise structure of those UNTFs?*

Second, it may be interesting to better understand the entrywise structure of the UNTFs already constructed in this note, for various ETFs more complicated than the simplex ETF.

QUESTION 6.2. *Let $\boldsymbol{M}$ be the Gram matrix of a UNTF lifted from some real ETF with Gram matrix $\boldsymbol{X}$.*

1. *How many distinct angles (magnitudes of inner products) are there in $\boldsymbol{M}$? What properties of $\boldsymbol{X}$ does this quantity depend on?*
2. *Are the entry graphs of $\boldsymbol{M}$ as described in Section 5 indeed always regular for certain families of ETFs? Do they enjoy any other structure in general or for particular families of ETFs?*
3. *The complete graph on N vertices is associated in a natural way to the simplex ETF as the "sign graph," in which vertices are indexed by ETF vectors and are connected if their inner product is negative. The Johnson graph of order N, associated per Section 4 to the lifted UNTF, happens to be the* line graph *of the complete graph—its vertices may be identified with edges of the complete graph, and its edges describe the incidences of edges in the complete graph. Do the entry graphs of lifted UNTFs in general have an analogous interpretation?*

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